Mathematical Methods in Computer Science

Exercise 3

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1. **The Probabilistic Method:**

   (a) Let $\mathcal{F}$ be a family of subsets of $N = \{1, ..., n\}$, and suppose there are no $A, B \in \mathcal{F}$ satisfying $A \subset B$. Let $\sigma \in S_n$ be a random permutation of the elements of N and consider the random variable

   $$X = |\{i \ : \ \{\sigma(1), \sigma(2), ..., \sigma(i)\} \in \mathcal{F}\}|.$$

   By considering the expectation of X prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. (Hint: consider $X = \sum_{A \in \mathcal{F}} X_A$, where $X_A(\sigma) = 1$ iff $\{\sigma(1), ..., \sigma(i)\} \in A$).

   (b) A set of positive integers $x_1, ..., x_k$ is said to have distinct sums if all sums $\sum_{i \in S} x_i, \quad S \subset \{1, ..., k\}$ are distinct. Let $f(n)$ be the maximal k for which exists a set $x_1, ..., x_k \subset \{1, ..., n\}$ with distinct sums.

   i. Prove: $f(n) \geq 1 + \lfloor \log_2(n) \rfloor$.

   ii. Prove: $f(n) < \log_2(n) + \log_2 \log_2(n) + O(1)$.

   iii. Fix $\{x_1, ..., x_k\} \subset \{1, ..., n\}$ with distinct sum. Let $\epsilon_1, ..., \epsilon_k$ be independent with $Pr(\epsilon_i = 0) = Pr(\epsilon_i = 1) = 1/2$ and set $X = \sum_{i=1}^{k} x_i \epsilon_i$. Use Chebyschev's inequality to prove:

   $$f(n) \leq \log_2 n + \frac{1}{2} \log_2 \log_2 n + O(1)$$

2. **Measure Concentration - A simple version of Chernoff bound:**
   Suppose we sample from a source of wisdom $6n$ times. Each time we ask the same YES or NO question and have a 3/4 chance of getting a correct answer (there are no correlations between the trials). If we take the majority of the $6n$ answers, show that we get the wrong answer with probability less than $1/2^n$. Hint: It is possible to negotiate this calculation using only basic arithmetic and the fact that the number of subsets of a set of size $6n$ is 2^{6n}. Of course you can use stronger machinery.

3. **Lagrange Multipliers**:

   (a) Find the minimum of $f(x) = \sum_{i=1}^{n} x_i^2$ subject to $\sum_{i=1}^{n} x_i = 1$.

   (b) Find the minimum of $f(x) = \sum_{i=1}^{n} x_i^2$ subject to $x^\top Q x = 1$ for positive definite matrix Q.

   (c) Verify Schwartz inequality $< x, y > \leq \ ||x|| \cdot ||y||$ for $x, y \in \mathbb{R}^n$ by solving $\max < x, y >$ subject to $||x|| = ||y|| = 1$.