Mathematical	Methods	in	Computer	Science

Exercise 2

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1. **Convexity:** A set C is called convex if for every $x, y \in C$ the segment between them is included in C, i.e., $\lambda x + (1-\lambda)y \in C$ for every $\lambda \in [0,1]$. A function $f : \mathbb{R} \to \mathbb{R}$ is called convex if the set above its graph $\{(x,z) : \ z \geq f(x)\}$ is a convex set.

   (a) Show that $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if for every $x, z \in \mathbb{R}$ and $\lambda \in [0,1]$: $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$.

   (b) Prove the Jensen inequality: For every convex function $f : \mathbb{R} \to \mathbb{R}$, and for every set of points $x_i \in \mathbb{R}$ and probability vector $p_1, ..., p_n$: $f(\sum_{i=1}^n p_i x_i) \leq \sum_{i=1}^n p_i f(x_i)$.

   (c) Prove that a continuously twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if $f"(x) \geq 0$ for all $x \in \mathbb{R}$.

   (d) A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for every $x, z \in \mathbb{R}^n$ and $\lambda \in [0,1]$: $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$. Prove that a continuously twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the matrix of second derivatives $H_{ij} = \frac{\partial f}{\partial x_i \partial x_j}$ is positive semi definite. The matrix H is called the Hessian of f.

2. **Basic Feasible Solutions in LP** Given $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ where $m \leq n$, define $P = \{x \in \mathbb{R}^n | x \geq 0, Ax = b\}$ to be the set of feasible solutions. We assume thamt A has rank m. Let $A_i \in \mathbb{R}^m$ be the i'th column of A. Consider a set $I \subseteq \{1, ..., n\}$ of m elements and the $m \times m$ matrix A_I of rank m whose columns are $(A_i)_{i \in I}$. We consider the unique vector x such that $x_I = A_I^{-1} b$ and $x_{\overline{I}} = 0$ where $\overline{I} = \{1, ..., n\} \setminus I$. If $x \in P$, i.e., $x \geq 0$, say that x is a *Basic Feasible Solution*. The simplex algorithm iterates on the set of the basic feasible solutions, therefore its run-time is at most $\binom{n}{m}$ which may however be exponential in n.

   (a) A point z in a convex set C is called extreme point if there are no two point $x \neq y \in C$ such that $z = \lambda x + (1-\lambda)y$ for $\lambda \in (0,1)$. Describe the extreme points of the convex set $C = \{x \in \mathbb{R}^n | x \geq 0, \sum_{i=1}^n x_i = 1\}$.

   (b) Prove that $P = \{x \in \mathbb{R}^n | x \geq 0, Ax = b\}$ is a convex set. Prove that $x \in P$ is a basic feasible solution if and only if x is an extreme point in P.

   (c) Prove that if the linear program $max < c, x >$ s.t $x \in P$ has an optimal solution then it has an optimal solution which is a basic feasible solution.

   (d) **Integral Solutions:** Conclude that if all the sub matrices of A have determinant $0, 1, -1$ and the vector $b \in \mathbb{Z}^m$ then there is an optimal solution to this linear program with integer entries. We call a matrix a matrix A with this property *totally unimodular.*

3. **Min-Cut Max-Flow**: A *flow network* is a directed graph $G = (V, E)$ together with a capacity function $c : E \to \mathbb{R}^{\geq 0}$ and two distinguished vertices $s, t \in V$, where vertex s

is called the source and vertex t is called the sink. For every $v \in V$, let $IN(v) \stackrel{def}{=} \{e \in E : e = (u,v) \text{ for some } u \in V\}$ and $OUT(v) \stackrel{def}{=} \{e \in E : e = (v,u) \text{ for some } u \in V\}$. A flow in G is a function $f : E \to \mathbb{R}$ which satisfies the following three conditions:

- Non-negativity: for all $e \in E,\ f(e) \geq 0$.
- Capacity Constraints: for all $e \in E, f(e) \leq c(e)$.
- Flow conservation: for all $v \in V \setminus \{s,t\}$ $\sum_{e \in IN(v)} f(e) = \sum_{e \in OUT(v)} f(e)$.

(a) Write the linear program "$max - flow$" that maximizes the total flow out of the vertex s, namely $\sum_{e \in OUT(s)} f(e) - \sum_{e \in IN(s)} f(e)$. We denote this quantity by $\phi(f)$. The linear program should have a polynomial number (in the size of the graph G) of constraints.

(b) Prove that if the capacities are integral, i.e., $c \subset \mathbb{N} \cup \{0\}$ then there is an optimal flow with $f(e)$ integral for every $e \in E$. (Hint: Convert the LP problem to have the form max $< c, x >$ s.t. $Ax = b, x \geq 0$, and show that A is totally unimodular (T.U.M) as in [2d]. Prove the following fact: Let $G = (V,E)$ be a directed graph and $M \in \mathbb{R}^{|V| \times |E|}$ its adjacency matrix, i.e, for an edge $e = (v,u)$ then $M(v,e) = 1$ and $M(u,e) = -1$, then M is T.U.M. Prove this fact by induction on the size of M. Considering three cases — (i) there is a column with all zeros, (ii) there is a column with a single 1 or a single -1, (iii) Every column has a 1 and a -1).

(c) Write the dual linear program of "$max - flow$".

(d) A cut in G is a partition of the vertex set V into disjoint sets S and T such that $S \cup T = V$, where $s \in S$ and $t \in T$. The capacity C of a cut (S,T) is $C(S,T) \stackrel{def}{=} \sum_{e \in E(S,T)} c(e)$, where $E(S,T) \stackrel{def}{=} \{(u,v) \in E :\ u \in S \text{ and } v \in T\}$. Denote the set of all cuts in G by $C(G) \stackrel{def}{=} \{(S,T) :\ (S,T) \text{ is a cut in } G\}$. Prove the theorem: *The maximum of $\phi(f)$ over all flows f in G is equal to the smallest capacity $C(S,T)$ of any cut in G.*

(e) Another way to formulate the "max-flow" problem as a linear program is via flow decomposition. A path p from s to t of length m is a sequence of distinct vertices $s = v_{p_1}, ..., v_{p_m} = t$ which starts at s and ends in t. We consider the set $P = \{p = (v_{p_1} = s, ..., v_{p_{length(p)}} = t)\}$ of all the s to t paths in the graph G. We denote the flow along the path p by f_p, i.e., along every edge e in the path p a f_p flow is added (note that by using this formulation we need not take care of the flow-conservation). The vector $(f_p)_{p \in P}$ describes a valid flow in the graph G and the maximum flow problem takes the form:

$$\begin{array}{c} \max \sum_{p \in P} f_p \\ \forall e \in E \quad \sum_{\text{paths } p \text{ through } e} f_p \leq c(e) \\ \forall p \in P \quad f_p \geq 0 \end{array}$$

i. Describe a graph $G = (V,E)$ such that the above linear program has exponential size (in the size of G).

ii. Write the dual program.

iii. Conclude the max-flow min-cut theorem from the LP duality applied to this linear program.

4. **Von-Neumann Min-Max theorem**: Let $A \in \mathbb{R}^{m \times n}$ be a payoff table of a two preson game. Recall that p is called probability vector if it is non-negative and its entries sum up to unity. Prove that x, y:

$$\max_x \min_y y^\top A x = \min_y \max_x y^\top A x$$

where the max and the min are taken for x, y proabbility vectors. The above quantity is called the *value* of the above game.