Mathematical Methods in Computer Science	
Exercise 1	
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1. **Matrix Norms** Prove:

   (a) $||A||_{1\to 1} = \max_j \sum_i |A_{ij}|$

   (b) $||A||_{\infty\to\infty} = \max_i \sum_j |A_{ij}|$

   (c) Let $A \in \mathbb{R}^{n\times m}$.

      i. Show that $||Ax||_{\infty\to 1} = \max_{||x||_\infty=1} \frac{||Ax||_1}{||x||_\infty} = \max_{x\in\{-1,1\}^n, y\in\{-1,1\}^m} \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}x_iy_j$.

      ii. Let $||A||_C = \max_{I\subseteq[n],J\subseteq[m]} |\sum_{i\in I}\sum_{j\in J} a_{ij}|$ be the cut norm of the matrix A. Show that $||A||_C \le ||A||_{\infty\to 1} \le 4||A||_C$.

2. Let A be a Hermitian matrix. Show that $rank(A) \ge \frac{trace(A)^2}{trace(A^2)}$ (Hints: Compute the trace and use Cauchy-Swartz inequality).

3. Let $A \in R^{m\times n}$ where $n \ge m$. Show that the m largest eigenvalues of $A^\top A$ and $AA^\top$ are the same. Show that the rest of the eigenvalues of $A^\top A$ are zero.

4. **The Friendship Theorem:** Suppose G is a graph in which any two vertices have exactly one common neighbor. Then there is a vertex which is adjacent to any other vertex. We prove this theorem along the following lines:

   (a) Exhibit a graph with this property.

   (b) Suppose the assertion is false. Show that if G is counterexample then G is d-regular (meaning every vertex has degree d).

   - Let u,v be non-adjacent vertices. Let $w_1,...,w_{d_u}$ be the neighbors of u and $z_1,...,z_{d_v}$ be the neighbors of v. Let $w_1 = z_1$ be their common neighbor. w_1 is adjacent to one of the vertices $w_2,...,w_{d_u}$ (why?). z_1 is adjacent to one of the vertices $z_2,...,w_{d_v}$ (why?). Assume that $w_1 \sim w_2$ (w_1 is adjacent to w_2) and $z_1 \sim z_2$. Show that for $i = 3,...,d_u$ $w_i \sim z_{w_i}$ (z_{w_i} is a node from $z_1,...,z_{d_v}$) and all the z_{w_i} are different. Conclude that $d_v \ge d_u$ and that $d_u = d_v$.
   - Show that the triangle graph is the only possible solution 2-regular graphs. For the rest of the proof we assume $d > 2$.
   - Use non-adjacency to extend this to every vertex in G.
   - Show that the number of vertices $n = d^2 - d + 1$.

- Let A be the adjacency matrix of the graph G. Use the eigenvalues of A and A^2 to show that $\sqrt{d-1} = \frac{d}{n_1 - n_2}$ for some integers n_1, n_2. Prove that if $\sqrt{d-1}$ is a rational number then it is an integer.
- Show that $\sqrt{d-1}$ divides d and divides $d-1$ and conclude that $d = 2$.

5. **Matrix Properties** Show the following claims:

   (a) A is Hermitian if and only if $< Ax, x > \in R$ for every $x \in \mathbb{C}^n$.

   (b) Hermitian matrix A is PSD (Positive Semi Definite) if and only if $< Ax, x > \geq 0$ for every $x \in \mathbb{C}^n$.

   (c) A function $B(x, y)$ is an inner product iff there is a PD (Positive Definite) matrix such that $B(x, y) = < Ax, y >$.

   (d) Let $U \in M_n(\mathbb{C})$. Show that the following statements are equivalent:

   - U is unitary, i.e., $U^\star U = I$.
   - $UU^\star = I$.
   - The columns of U form an orthogonal set.
   - For all $x \in \mathbb{C}^n$ the length $||x||_2$ is equal to the length of $||Ux||_2$.