Data Structures – LECTURE 2

Elements of complexity analysis

- Performance and efficiency
- Motivation: analysis of Insertion-Sort
- Asymptotic behavior and growth rates
- Time and space complexity
- Big-Oh functions: O(f(*n*)), Ω(f(*n*)), Θ(f(*n*))
- Properties of Big-Oh functions

Performance and efficiency

- We can quantify the performance of a program by measuring its run-time and memory usage.

  It depends on how fast is the computer, how good the compiler

  → a very local and partial measure!
- We can quantify the efficiency of an algorithm by calculating its space and time requirements as a function of the basic units (memory cells and operations) it requires

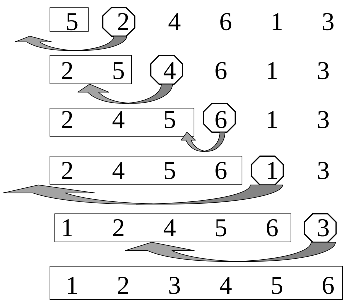
  → implementation and technology independent!

Example of a program analysis

Sort the array A of n integers

Insertion-Sort(*A*)	cost	times
1. for *j* ← 2 **to** *A.length*	*c1*	n
2. **do** *key* ← *A*[*j*]	*c2*	$n-1$
3. // Insert *A*[*j*] into *A*[1..*j*-1]		
4. *i* ← *j* – 1	*c4*	$n-1$
5. **while** *i* > 0 **and** *A*[*i*] > *key*	*c5*	$\sum_{j=2}^{n} t_j$
6. **do** *A*[*i*+1] ← *A*[*i*]	*c6*	$\sum_{j=2}^{n} (t_j - 1)$
7. *i* ← *i* – 1	*c7*	$\sum_{j=2}^{n} (t_j - 1)$
8. *A*[*i*+1] ← *key*	*c8*	$n-1$

Insertion-Sort example



Program analysis method

- The running time is the sum of the running times of each statement executed
- Each statement takes c_i steps to execute and is executed t_i times → total running time is

$$T(n) = \sum_{i=1}^{k} c_i t_i$$

Insertion-Sort analysis (1)

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

- Best case: the array is in sorted order, so $t_j=1$ for $j=2..n$; step 5 takes $\sum_{j=2}^{n} t_j = n$ and steps 6 and 7 are not executed. Thus

$$T(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$$

$T(n) = an + b$ *This is a linear function!*

Insertion-Sort Analysis (2)

- Worst case: the array is in reversed sorted order, so $t_j = j$ for $j = 2..n$, so step 5 takes

$$\sum_{j=2}^{n} j = n(n+1)/2 - 1$$

and steps 6 and 7 are always executed, so they take $\sum_{j=2}^{n} (j-1) = n(n+1)/2$

Overall

$$T(n) = (c_5 + c_6 + c_7)\frac{n^2}{2} + (c_1 + c_2 + c_4 + c_8 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2})n - (c_2 + c_4 + c_5 + c_8)$$

$T(n) = an^2 + bn + c$ *This is a quadratic function!*

Asymptotic analysis

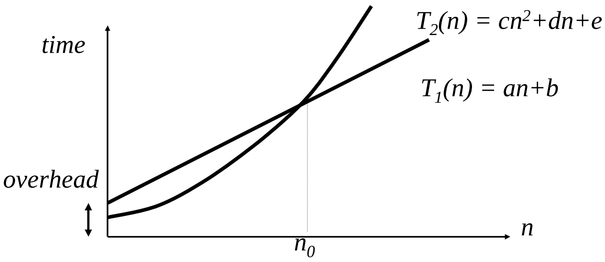
- We can write the running time of a program $T(n)$ as a function of the input size n:

$$T(n) = f(n)$$

- The function contains constants that are program and platform-dependent.
- We are interested in the asymptotic behavior of the program: how quickly does the running time grow as a function of the input size?

→ Rationale: if the input size n is small, all programs are likely to do OK. But they will have trouble when n grows. In fact, the program performance will be dominated by it!

Asymptotic analysis: best vs. worst case for Insertion-Sort



No matter what the constants are $T_2(n) > T_1(n)$ after a while

To summarize

- The efficiency of an algorithm is best characterized by its asymptotic behavior as a function of the input or problem size n.
- We are interested in both the run-time and space requirements, as well as the best-case, worst-case, and average behavior of a program.
- We can compare algorithms based on their asymptotic behavior and select the one that is best suited for the task at hand.

Time and space complexity

- Time complexity: $T(n)$

  How many operations are necessary to perform the computation as a function of the input size n.

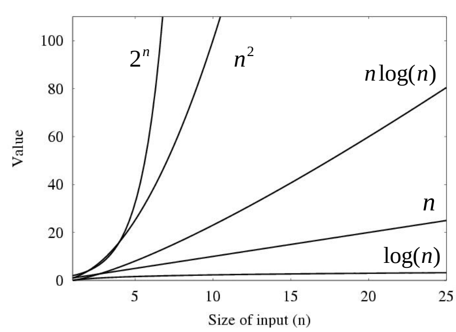
- Space complexity: $S(n)$

  How much memory is necessary to perform the computation as a function of the input size n.

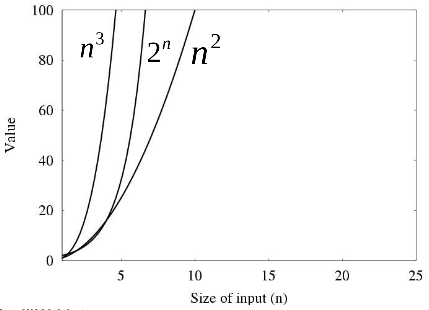
- Rate of growth:

  We are interested in how fast the functions $T(n)$ and $S(n)$ grow as a function of the input size n.

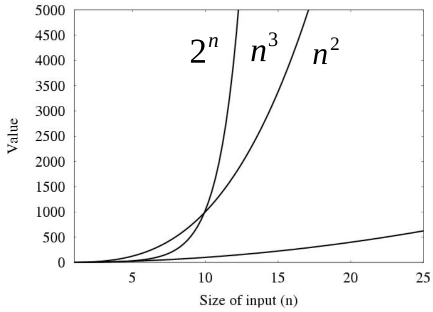
Rates of growth of common functions



Rates of growth: behavior (1)



Rates of growth: behavior (2)



Problem size as a function of time

Algorithm	**Time Complexity**	**Maximum problem size**		
		1 sec	**1 min**	**1 hour**
A_1	n	*1000*	*6 x10*^{4}	*3.6* x *10*^{6}
A_2	$n \log_2 n$	*140*	*4893*	*2.0* x *10*^{5}
A_3	n^2	*31*	*244*	*1897*
A_4	n^3	*10*	*39*	*153*
A_5	2^n	*9*	*15*	*21*

Assuming one unit of time equals one millisecond.

Effect of a tenfold speedup

Algorithm	**Time Complexity**	**Maximum problem size**	
		before speed-up	**after speed-up**
A_1	n	s_1	$10s_1$
A_2	$n \log_2 n$	s_2	*approx.* $10s_2$ *(for large* s_2*)*
A_3	n^2	s_3	$3.16s_3$
A_4	n^3	s_4	$2.15s_4$
A_5	2^n	s_5	$s_5 + 3.3$

Asymptotic functions

Define mathematical functions that estimate the complexity of algorithm A with a growth rate that is independent of the computer hardware and compiler. The functions ignore the constants and hold for sufficiently large input sizes n.

- **Upper bound** O(f(n)): <u>at most</u> f(n) operations
- **Lower bound** Ω(f(n)): <u>at least</u> f(n) operations
- **Tight bound** Θ(f(n)) : <u>order of</u> f(n) operations

Asymptotic upper bound – Big-Oh

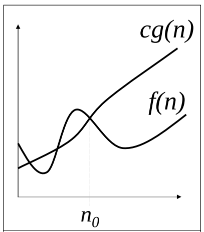
Let $f(n)$ and $g(n)$ be two functions from naturals to positive reals

$f(n) = O(g(n))$

if there exist $c > 0$ and $n_0 > 1$ such that

$$f(n) \leq c \times g(n)$$

for all $n \geq n_0$



f(n) =O(g(n))

Asymptotic lower bound – Big-Omega

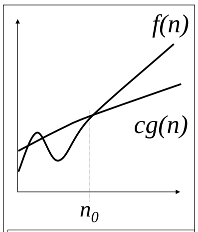
Let $f(n)$ and $g(n)$ be two functions from naturals to positive reals

$f(n) = \Omega(g(n))$

if there exist $c > 0$ and $n_0 > 1$ such that

$$f(n) \geq c \times g(n))$$

for all $n \geq n_0$



f(n) = Ω(g(n))

Asymptotic tight bound – Big-Theta

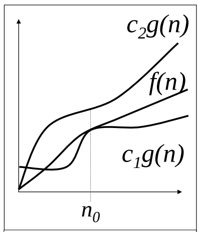
Let $f(n)$ and $g(n)$ be two functions from naturals to positive reals

$f(n) = \Theta(g(n))$

if there exist $c_1, c_2 > 0$ and $n_0 > 1$ such that

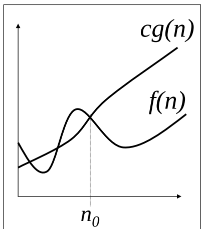
$$0 \leq c_1 \times g(n) \leq f(n) \leq c_2 \times g(n)$$

for all $n \geq n_0$

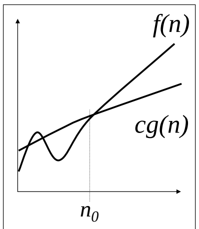


f(n) = Θ(g(n))
Tight bound

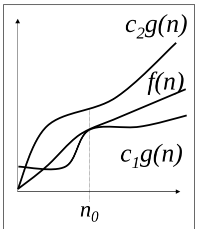
Graphs for O, Ω, and Θ



f(n) =O(g(n))
Upper bound



f(n) = Ω(g(n))
Lower bound



f(n) = Θ(g(n))
Tight bound

Properties of the O, Ω, and Θ functions

<u>Theorem</u>: f is tight iff it is an upper <u>and</u> a lower bound:

$f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

- <u>Reflexivity</u>:
  $f(n) = O(f(n))$; $f(n) = \Omega(f(n))$; $f(n) = \Theta(f(n))$
- <u>Symmetry</u>:
  $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$
- <u>Transitivity</u>:
  $f(n) = O(g(n))$ and $g(n) = O(h(n))$ then $f(n) = O(h(n))$
  $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ then $f(n) = \Omega(h(n))$
  $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ then $f(n) = \Theta(h(n))$

Properties of the O, Ω, and Θ functions

For O, Ω, and Θ:

- $O(O(f(n)) = O(f(n))$
- $O(f(n) + g(n)) = O(f(n)) + O(g(n))$
- $O(f(n).g(n)) = O(f(n)).O(g(n))$
- $O(\log n) = O(\lg n)$ — lg is $\log_2$
- Polynomials: $O(\sum_{i=1}^{k} a_i n^i) = O(n^k)$
- Factorials:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

$O(n!) = O(n^{n+0.5})$

$O(\log n!) = O(n \lg n)$

Asymptotic functions

- Asymptotic functions are used in conjunction with recurrence equations to derive complexity bounds
- Proving a lower bound for an algorithm is usually harder than proving an upper bound for it. Proving a tight bound is hardest!
- Note: still does not answer the if this is the <u>least</u> or the <u>most</u> work for the given problem. For this, we need to consider upper and lower problem bounds (later in the course).