CS294-1 On-line Computation & Network Algorithms	**Spring 1997**

Lecture 23: April 22

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This lecture continued the presentation of the results from [BBBT]. The results from [Bartal] are used to construct a randomized algorithm for the metrical task system (MTS) problem on a general metric space by using k-HST's to probabilistically approximate the metric space.

In the previous lecture, a problem similar to the MTS problem on a k-HST was defined, which we called Scenario 0. We consider a rooted tree with metric spaces numbered $1, \ldots, b$ as its children in which any two points in different child spaces are separated by a distance d. We assume that we already have algorithms for each of the child spaces with competitive ratios $r_1, \ldots, r_b$. Scenario 0 is then very close to MTS on this space, except that when an adversary pays some local cost σ_i in metric space i, an on-line algorithm is charged $r_i \sigma_i$.

Scenario 0 was then refined so that movement between states is charged unfairly as are local costs. This refined problem was termed Scenario 1 and includes a distance ratio s by which movement costs paid by an on-line algorithm are scaled. It was shown that a work function algorithm (termed Strategy 1) was $(r_1 + 2sb^{1/t}t)$-competitive for Scenario 1 where t is an odd integer. This algorithm together with the next lemma allow us to construct an algorithm for the general MTS problem.

Lemma 23.1 *Let A be an r-competitive algorithm for Scenario 1 with distance ratio s. There exists an algorithm $\hat{A}$ with distance ratio $\hat{s}$ that is r-competitive with amortized ratios r_i if $s \geq \hat{s} + \Phi_{\max}/d$. Furthermore, the potential function for $\hat{A}$ is bounded by $\Phi_{\hat{A}} \leq \Phi_A + \Phi_{\max}$.*

Proof: Let Φ_i be the potential function that amortizes the costs payed by the algorithm running within region i. $\hat{A}$ uses the same probability distribution (p_i) used by A.

$$\Phi_{\hat{A}} = \Phi_A + \sum_j p_j \Phi_j$$

When $\hat{A}$ receives a request (i, σ), A changes its probability distribution, in particular lowering the probability of being in state i from p_i to p_i'. The cost of movement for $\hat{A}$ is then

$$\begin{aligned}
c_{\hat{A}} &= (p_i - p_i')\hat{s}d + \Delta\Phi_{\hat{A}} \\
&= (p_i - p_i')\hat{s}d - (p_i - p_i')\Phi_i + \sum_{i \neq j}(p_j - p_j')\Phi_j \\
&\leq (p_i - p_i')\hat{s}d - (p_i - p_i')\Phi_i + (p_i - p_i')\Phi_{\max} \\
&\leq (p_i - p_i')(\hat{s}d + \Phi_{\max}) - (p_i - p_i')(\hat{s} + \Phi_{\max}/d)d \\
&\leq (p_i - p_i')sd = c_A
\end{aligned}$$

The local cost is

$$\begin{aligned}
c_{\hat{A}} &= p_i'(r_i\sigma - (\Phi_i' - \Phi_i)) + \Delta\Phi \\
&= p_i'(r_i\sigma - (\Phi_i' - \Phi_i)) + p_i'(\Phi_i' - \Phi_i) + \Delta\Phi_A \\
&= c_A + \Delta\Phi_A
\end{aligned}$$

Since the movement costs and local costs are both r-competitive when considered individually, $\hat{A}$ is r-competitive. ∎

Now we can use Strategy 1 to obtain a randomized algorithm for the MST problem on a k-HST (and thus on a general metric space).

Lemma 23.2 *Consider an n point k-HST with depth D and $k \geq 9D$. Then the competitive ratio of applying Strategy 1 with $\log_2 n \leq t < \log_2 n + 2$ is $\leq 9D \log_2 n$.*

If we let $k = 18 \log_2 \Delta$ this yields a competitive ratio of $O(\log n \log \Delta)$ for $\Omega(\log \Delta)$-HST's which can be used to obtain a competitive ratio of $O(\log_3 n \log^2 \Delta)$ for a general metric space.

Proof: Assume each subspace has diameter $\leq \Delta/k$, depth $D-1$, and a $(9(D-1)\log_2 n)$-competitive algorithm with $\Phi \leq 9(D-1)\Delta/k$. Then for a tree with diameter Δ and depth D, there is an algorithm with competitive ratio $\leq 9D \log_2 n$ with the potential function bounded by $\Phi \leq 9D\Delta$.

If v_i is in region i then $d(v_i, v_j) \geq \Delta - 2\Delta/k$. By letting $\hat{s} = \frac{k}{k-2}$ and $t = \log_2 n$, then $r = r_1 + 2sb^{1/t}t \leq 9(D-1)\log_2 n + 9 \log_2 n = 9D \log_2 n$. ∎

If the child regions in the k-HST are of the same size, then it is hard to improve on the results shown so far. However, the child regions may be of different sizes in which case Strategy 1 may not yield the best competitive ratio. For a simple k-HST with two child regions having competitive ratios r_1 and r_2, define Strategy 2 by letting

$$p_1 = \frac{e^{\frac{r_1 - r_2}{s}} - e^{\frac{r_1 - r_2}{s}\left(\frac{1}{2} + \frac{4}{2d}\right)}}{e^{\frac{r_1 - r_2}{s}} - 1}$$

Lemma 23.3 *The competitive ratio of Strategy 2 is*

$$c = r_1 + \frac{r_1 - r_2}{e^{\frac{r_1 - r_2}{s}} - 1}$$

Note that for two unbalanced regions with competitive ratios $\log n_1$ and $\log n_2$, this strategy allows us to achieve a competitive ratio of $\log(n_1 + n_2)$.

We can now create Strategy 3 by combining Strategy 1 and Strategy 2 as follows: (recall $r_1 \geq r_2 \cdots \geq r_b$) if $r_1 \leq 128 \log^2 n - 32 \log n$ then use Strategy 1 with t chosen as before $(2 \log_2 n \leq t < \log_2 n)$. Otherwisw, combine spaces 2 through b with Strategy 1 and then use Strategy 2 to combine space 1 with the resulting space. For appropriate choices of k and s, this strategy yields a competitive ratio $O(\log^2 n)$ and the general result (Theorem 22.1) follows.

References

[Bartal96] Y. Bartal. Probabilistic Approximation of Metric Spaces and its Algorithmic Applications. In *Proc. of the 37th Ann. IEEE Symp. on Foundations of Computer Science*, October 1996.

[BBBT97] Y. Bartal, A. Blum, C. Burch, and A. Tomkins. A polylog(n)-Competitive Algorithm for Metrical Task Systems . In *Proc. of the 29th Ann. ACM Symp. on Theory of Computing*, May 1997.