Advanced Algorithms – Exercise 3

Deadline: 6/1/2006

1. In order to run the Simplex algorithm, we need to have some initial base. In this question we will see how to find this initial base. As usual, we consider the following LP, denoted by LP_1 (A is an $m \times n$ matrix with degree m):

   Minimize: $c^T x$

   Subject to:

   - $Ax = b$
   - $x \geq 0$

   The idea is to add new variables, $x'_1, ..., x'_m$, and use the following LP, which we will denote as LP_2, to find a feasible base to LP_1:

   Minimize: $\Sigma_{i=1}^{m} x'_i$

   Subject to:

   - $Ax + Ix' = b$
   - $x, x' \geq 0$

   where I is the $m \times m$ unit matrix.

   (a) Prove that the optimal solution of LP_2 is 0 if and only if LP_1 is feasible.

   (b) To know if the optimal solution of LP_2 has a value of 0, we need to solve LP_2. For that, we need an initial base for LP_2. Find such an initial base.

   (c) Using the optimal solution of LP_2, find an initial base for LP_1.

2. In the *maximum weighted matching* problem, we are given an undirected graph $G = (V, E)$, with weights w_e on the edges. The goal is to find a maximum weighted matching. That is, to find a set M of edges of maximal weight, such that no two edges of M share the same vertex. This question devises a primal-dual 2-approximation algorithm for this problem. Consider the following LP formulation of the problem.

   Maximize: $\Sigma_{e \in E} w_e x_e$

   Subject to:

   - for each $v \in V$: $\Sigma_{e|v \in e} x_e \leq 1$
   - for each $e \in E$: $x_e \geq 0$

   (a) Describe the dual program for the LP.

   (b) Specify the complementary slackness conditions. State the conditions in words.

(c) Show a 2-approximation algorithm, based on the primal-dual schema, for the maximum weighted matching problem.

Remark: In fact, this problem can be solved in polynomial time. Moreover, it is known that the optimal fractional solution is equal to the optimal integral solution.

3. Consider a variation on the MAX-CUT problem, Where we are given two sets S_1 and S_2, both consist of pairs of vertices. The additional constraint is that the pairs in S_1 must be in the same side of the cut, and pairs in S_2 need to be separated. You may assume that a feasible solution exists.

   (a) Give a strict quadratic program and a vector program relaxations for this problem.

   (b) Use the SDP-based MAX-CUT algorithm taught in class to derive an approximation algorithm for our variation that provides the same approximation ratio.