# An analysis of the Shapley value and its uncertainty for the voting game

Shaheen Fatima
Department of Computer Science
University of Liverpool
Liverpool L69 7ZF, U.K.
shaheen@csc.liv.ac.uk

Michael Wooldridge
Department of Computer Science
University of Liverpool
Liverpool L69 7ZF, U.K.
mjw@csc.liv.ac.uk

Nicholas R. Jennings
School of Electronics and Computer Science
University of Southampton
Southampton SO17 1BJ, U.K.
nrj@ecs.soton.ac.uk

## ABSTRACT

The Shapley value provides a unique solution to coalition games and is used to evaluate a player's prospects of playing a game. Although it provides a unique solution, there is an element of uncertainty associated with this value. This uncertainty in the solution of a game provides an additional dimension for evaluating a player's prospects of playing the game. Thus, players want to know not only their Shapley value for a game, but also the associated uncertainty. Given this, our objective is to determine the Shapley value and its uncertainty and study the relationship between them for the voting game. But since the problem of determining the Shapley value for this game is #P-complete, we first present a new polynomial time randomized method for determining the approximate Shapley value. Using this method, we compute the Shapley value and correlate it with its uncertainty so as to allow agents to compare games on the basis of both their Shapley values and the associated uncertainties. Our study shows that, a player's uncertainty first increases with its Shapley value and then decreases. This implies that the uncertainty is at its minimum when the value is at its maximum, and that agents do not always have to compromise value in order to reduce uncertainty.

## 1. INTRODUCTION

The Shapley value provides a *unique* solution to coalition games and is therefore used to evaluate a player's prospects of playing a game. Although the Shapley value provides a unique solution, it has two key drawbacks. First, for the weighted voting game that we consider, the problem of determining the Shapley value is #P-complete [1]. Second, it provides the solution only with a limited degree of certainty [4]. Thus the uncertainty in the Shapley value provides an additional dimension for evaluating a player's prospects of playing a game and a measure of uncertainty would serve as a useful tool to investigate this aspect of a game. Characterizing a game by both its value and uncertainty is like characterising a weapon by its power and precision, or a financial stock by its expected return and risk [2].


*AAMAS'05* July 25-29, 2005, Utrecht, Netherlands


Roth showed that the Shapley value of a game equals its utility, if and only if the underlying player preferences are neutral to both *ordinary*[1] and *strategic* risk [3, 4]. Otherwise, the Shapley value is not the same as utility and is therefore insufficient for decision-making purposes. Kargin extended this concept further by introducing a measure for determining the strategic risk [2]. This measure is called the *uncertainty* of the Shapley value and it provides a yardstick for quantifying the strategic risk. Thus, in order for a player to make more informed decisions, it is important for it to not only know its Shapley value, but also the relation between this value and its uncertainty. However, to date, there has been no analysis of this relationship.

Given this, our objective is to analyse the relation between the Shapley value and its uncertainty for the *voting game*. However, since the problem of determining the Shapley value has been shown to be #P-complete [1], we present a new *randomised* method (that has polynomial time complexity) for computing the *approximate* Shapley value. Using this method, we determine the Shapley value and correlate it with its uncertainty.

Section 2 defines the Shapley value and its uncertainty. Section 3 describes the weighted voting game. Section 4 shows the relation between the Shapley value and its uncertainty. Section 5 concludes.

## 2. THE SHAPLEY VALUE AND ITS UNCERTAINTY

We begin by introducing coalition games and then define the weighted voting game. Coalition games are of two types: those with *transferable payoff* and those with *non-transferable payoff*. A coalition game with transferable payoff, $\langle N, v\rangle$, consists of a finite set ($N = \{1, 2, \dots, n\}$) of players and a function ($v$) that associates with every non-empty subset $S$ of $N$ (i.e., a *coalition*) a real number $v(S)$ (the worth of $S$).

For each coalition $S$, the number $v(S)$ is the total payoff that is available for division among the members of $S$ (i.e., the set of joint actions that coalition $S$ can take consists of all possible divisions of $v(S)$ among the members of $S$). Coalition games with non-transferable payoffs differ from ones with transferable payoffs in the following way. For the former, each coalition is associated with a *set* of payoff vectors that is not necessarily the set of all possible divisions of some fixed amount. In this paper, we focus on the Shapley value for a game with transferable payoffs.

[1]Ordinary risk involves the uncertainty that arises from the chance mechanism involved in lotteries. On the other hand, *strategic risk* involves the uncertainty that arises as a result of interaction in a game of strategic players (i.e., those players that are not dummy).

Let $\mathcal{S}$ denote the set $N - \{i\}$ and $f_i : \mathcal{S} \rightarrow 2^{N-\{i\}}$ be a random variable that takes its values in the set of all subsets of $N - \{i\}$, and has the probability distribution function ($g$) defined as:

$$g\{f_i(\mathcal{S}) = S\} \quad = \quad \frac{|S|!(n - |S| - 1)!}{n!}$$

The random variable $f_i$ is interpreted as the random choice of a coalition that player $i$ joins. A player's Shapley value [5] is defined in terms of its *marginal contribution*. The marginal contribution of player $i$ to coalition $S$ with $i \notin S$ is a function $\Delta_i v$ that acts as: $\Delta_i v(S) = v(S \cup \{i\}) - v(S)$

DEFINITION 1. *The Shapley value ($\varphi_i$) of the game $\langle N, v\rangle$ for player $i$ is the expectation (E) of its marginal contribution to a coalition that is chosen randomly, i.e., $\varphi_i(N, v) = E\{\Delta_i v \circ f_i\}$*

The Shapley value is interpreted as follows. Suppose that all the players are arranged in some order, all orderings being equally likely. Then $\varphi_i(N, v)$ is the *expected marginal contribution*, over all orderings, of player $i$ to the set of players who precede him. The *uncertainty* of the Shapley value, is defined as follows [2]:

DEFINITION 2. *The uncertainty ($\beta_i$) for player $i$ is the variance ($Var$) of its marginal contribution. Thus $\beta_i(N, v) = Var\{\Delta_i v \circ f_i\}$*

Thus, while a player's Shapley value is the expectation (i.e., the mean), its uncertainty is the variance (i.e., the square of the standard deviation) of its marginal contribution. In other words, the uncertainty is the expectation of the squared difference between the actual and expected marginal contributions.

The utility of a player that is not neutral to strategic risk depends on both its Shapley value and the associated uncertainty. Furthermore, such a player's utility function is subjective and different players may have different functions for the same game. But for a given game, the relation between the Shapley value and its uncertainty is not subjective to player preferences and is the same for all players. We therefore analyse this relation for the voting game described in Section 3.

# 3. THE WEIGHTED VOTING GAME

There is a set of $n$ players that may, for example, represent shareholders in a company or members in a parliament. The weighted voting game is a game $G = \langle N, v\rangle$ in which

$$v(S) = \begin{cases} 1 & \text{if } w(S) \geq q \\ 0 & \text{otherwise} \end{cases}$$

for some $q \in I\!\!R_+$ and $w \in I\!\!R_+^N$, where $w(S) = \sum_{i \in S} w_i$ for any coalition $S$. Thus $w_i$ is the number of votes that player $i$ has and $q$ is the number of votes needed to win the game (i.e., the *quota*). For this game (denoted $\langle q; w_1, \ldots, w_n\rangle$), a player's marginal contribution is either zero or one.

For this game, we present a randomised method for computing the Shapley value. Using this method we find the Shapley value and correlate it with its uncertainty. The analysis for a simple game is explained the following section.

# 4. SHAPLEY VALUE VS. UNCERTAINTY

Consider the game $\langle q; j, \ldots, j\rangle$ with $m$ parties. Each party has $j$ seats. If $q \leq j$, then there would be no need for players to form a coalition. On the other hand, if $q = mj$ ($m = |N|$ is the number of players), only the grand coalition is possible. Thus, the quota ($q$) satisfies the constraint: $(j + 1) \leq q \leq j(m - 1)$. A majority

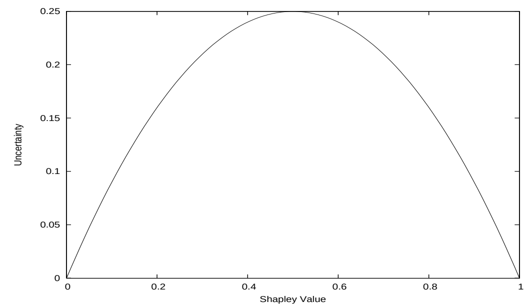


**Figure 1: Shapley value vs. uncertainty**

is decisive. The value of a coalition is one if the weight of the coalition is greater than or equal to $q$, otherwise it is zero.

Let $\varphi$ denote the Shapley value for a player and $\beta$ denote its uncertainty. Consider any one player. This player can join a coalition as the $i$th member where $1 \leq i \leq m$. However, the marginal contribution of the player is 1 only if it joins a coalition as the $\lceil q/j \rceil$th member. In all other cases, its marginal contribution is zero. Thus, the Shapley value for each player is $\varphi = 1/m$. We know from Definition 2, that a player's uncertainty is the variance of its marginal contribution. Hence, for each player, the uncertainty (denoted $\beta$) is:

$$\beta \quad = \quad (1 - \varphi)\varphi^2 + (1 - \varphi)^2\varphi \qquad (1)$$

Having expressed a player's uncertainty in terms of its Shapley value, we can now correlate them. To this end, Figure 1 shows how the uncertainty varies with the Shapley value. The same analysis is done for the more general case where all players do not have equal weight.

# 5. CONCLUSIONS

The uncertainty of the Shapley value is an additional dimension that an agent should take into account for evaluating its prospects of playing a game. This paper has analysed the relation between the Shapley value and its uncertainty for the weighted voting game. Since the problem of determining the Shapley value is #P-complete, we first presented a randomised method with polynomial time complexity. Using this method, we computed the Shapley value and correlated it with its uncertainty. Our study shows that a player's uncertainty first increases with its Shapley value and then decreases.

# 6. REFERENCES


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