# PGM 2002/03

# Langrage Multipliers

# The Lagrange Multipliers

The popular Lagrange multipliers method is used to find extremum points of a function on a boundary. In simpler words: we want to find maxima/minima of a function $\mathbf{f(X_1 \ldots X_n)}$ given some constraint/s $\mathbf{q_1(X_1 \ldots X_n)=0}$, $\mathbf{q_2(X_1 \ldots X_n)=0}$…

This can be solved by writing the Lagrangian:

$$L(X_1 \ldots X_n) = f(X_1 \ldots X_n) - \sum_i \lambda_i q_i(X_1 \ldots X_n)$$

and then solving a set of equations. $\lambda_l$ are called the **Lagrange Multipliers**.

# The Lagrange Equations

The equations we have to solve are:

$$\frac{\partial L(X_1 \ldots X_n)}{\partial X_1} = 0, \frac{\partial L(X_1 \ldots X_n)}{\partial X_2} = 0, \cdots, \frac{\partial L(X_1 \ldots X_n)}{\partial X_n} = 0$$

along with:

$$\frac{\partial L(X_1 \ldots X_n)}{\partial \lambda_1} = q_1(X_1 \ldots X_n) = 0, \cdots, \frac{\partial L(X_1 \ldots X_n)}{\partial \lambda_n} = 0$$

which gives us a set of n+k equations with n+k unknowns.

# Lagrange Multipliers: Intuition

Let us look at the simple case of **f(x,y)** [**f** for simplicity] and one constraint **q(x,y)=0** [**q** for simplicity] **(**mathematician here would of course request that the Lagrangian is "more or less" continuous and is differential in all variables). **p** is the maximum point of **f** on the surface (or manifold) defined by **q**.

- Since **p** is maximum, $\nabla$**f(x,y)** at point p (on the intersection with **q**) is at right angles to **q** or we would be able to move a bit on **q** and get a more extreme value of **f**.
- Since **q** is a constant value of **q(x,y)**, $\nabla$**q(x,y)** at point p is clearly at right angles to **q**.

From the above two point we deduce that $\nabla$**f(x,y)** and $\nabla$**q(x,y)** are on the same plane [coplanar].

# Lagrange Multipliers: Intuition (2)

A simple reminder from basic vector algebra: If two vectors are on the same plane then there must exist coefficients that satisfy:

$$a\mathbf{V_1} + b\mathbf{V_2} = \mathbf{0}$$

This can we easily expanded to any any number of vectors in any dimension and thus we can write:

$$\lambda\, \nabla \mathbf{f(x_1,..,X_n)} + \lambda_1\, \nabla \mathbf{q_1(x_1,..,X_n)} + \mathbf{\ldots} + \lambda_k\, \nabla \mathbf{q_k(x_1,..,X_n)} = 0$$

and since the $\lambda$'s are yet undetermined we can rewrite:

$$\nabla \mathbf{f(x_1,..,X_n)} - [\, \lambda_1\, \nabla \mathbf{q_1(x_1,..,X_n)} + \mathbf{\ldots} + \lambda_k\, \nabla \mathbf{q_k(x_1,..,X_n)}\, ] = 0$$

and explicitly writing the derivative for each unknown gives us the desired equations (along with the constraints). This system has one solution (excluding degenerative cases) and thus is the correct one.

# Lagrange Multipliers: Example

*Find the critical points of $f(x,y,z) = X^2-Y^2+2Z^2$ on the curve $xyz-1=0$ and $X^2+4Y^2+Z^2=1$*

Solution:

- $\nabla \mathbf{f} = [\ 2x\ ,\ -2y\ ,\ 4z\ ]$
- $\nabla \mathbf{q_1} = [\ yz\ ,\ xz\ ,\ xy]$
- $\nabla \mathbf{q2} = [\ 2x\ ,\ 4y\ ,\ 2z\ ]$

this gives us the equations:

(1) $2x - \lambda_1 yz + \lambda_2 2x = 0$

(2) $-2y - \lambda_1 xz + \lambda_2 4y = 0$

(3) $4z - \lambda_1 xy + \lambda_2 2z = 0$

along with

(4) $xyz-1=0$

(5) $X^2+4Y^2+Z^2=1$

which is now just tiresome to solve.