1 Matroids and the greedy algorithm

We had some problems in class with the proof of the 'greedy algorithms work if and only if there is a matroid structure on the solution set' theorem. As usual the source of the problem is in sloppy definitions.

Here we try to do it right.

Definition 1.1: A family of subsets $\mathcal{F}$ of $\{1...n\}$ is *hereditary* if for any two subsets A and B of $\{1...n\}$ if: $B \in \mathcal{F}$, and $A \subseteq B$ then $A \in \mathcal{F}$. ▮

Remark 1.2: In particular $\emptyset$ is in $\mathcal{F}$. (We will assume $\mathcal{F}$ to be non-empty). ▮

Definition 1.3: A non-empty hereditary family of subsets $\mathcal{F}$ of $\{1...n\}$ is a *family of independent sets of a matroid* iff it satisfies the *exchange property*: for any $A, B \in \mathcal{F}$ if $|B| > |A|$ then there is an element $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$. ▮

Remark 1.4: Observe that in class there were two inaccuracies which are resolved in this definition. First: an 'exchange' element b comes from $B \setminus A$. Second: the family $\mathcal{F}$ is assumed to be hereditary from the beginning, so the only additional property required for the matroidal structure is the exchange lemma. ▮

An algorithmic problem

Input A non-empty hereditary family of $\mathcal{F}$ of $\{1...n\}$ (the legal solutions of the problem); A nonnegative weight function μ on $\{1...n\}$. This weight function is extended to subsets of $\{1...n\}$ by taking $\mu(A) = \sum_{a \in A} \mu(a)$.

Output An element of $\mathcal{F}$ (i.e. a subset of $\{1...n\}$) of maximal weight.

We attempt to solve this problem using a *greedy algorithm*.

Definition 1.5: *A greedy algorithm* $\mathcal{A}$ for the above problem proceeds as follows to return a solution S.

1. Initialize $S = \emptyset$.
2. Proceed while you can:
   Choose an element $x \in \{1...n\} \setminus S$ of maximal (positive) weight such that $S \cup \{x\}$ is in $\mathcal{F}$; Set $S := S \cup \{x\}$.
3. If no such x is found in the second step: there are no elements $x \in \{1...n\} \setminus S$ such that $\mu(x) > 0$ and $S \cup \{x\}$ is in $\mathcal{F}$ - terminate and output S.

▮

Remark 1.6: Note that this is not the most general form of a greedy algorithm. In fact we have seen in class greedy algorithms which do not conform to this definition - for instance for the activity selection problem (SHIBUTZ MESIMOT).

Some more details. In the activity selection problem the family $\mathcal{F}$ of solutions is a non-empty hereditary family of subsets of $\{1...n\}$ (indices of non-intersecting time intervals). There is a weight function μ on $\{1...n\}$, giving a unit weight to each element. It is extended to $\mathcal{F}$ as above namely the weight of each element of $\mathcal{F}$ is its cardinality. We want to find an element of $\mathcal{F}$ of maximal weight. So far everything is as above. But: $\mathcal{F}$ does **not** have a matroid structure.Therefore the greedy algorithm that works for this question has to be more sophisticated. It does not choose any maximal weight (namely *any* since all the weights are the same) element that can be added to a current solution, without violating the constraints (staying in $\mathcal{F}$) - the new element has to optimize a local condition: minimal termination time. Observe also that the algorithm won't work for any weight function μ (verify!). ∎

We now claim that:

Theorem 1.7: *A greedy algorithm (as defined above) returns an optimal solution for any weight function iff $\mathcal{F}$ has the structure of a family of independent sets of a matroid.*

Proof: If $\mathcal{F}$ has the structure of a family of independent sets of a matroid then the greedy algorithm works - as we have seen in class for the problem of a maximal-weight linearly independent set of vectors.

In the other direction, if $\mathcal{F}$ does not have the structure of a family of independent sets of a matroid, this means that it does not satisfy the exchange property (since it is assumed to be hereditary), so there are sets $A, B \in \mathcal{F}$ with $|B| > |A|$, but with no element $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$. Now we can define (as we did in class) a weight function μ for which the greedy algorithms does not return an optimal solution.

Recall that we have set $\mu(a) = 1$ for any $a \in A$, $\mu(b) = 1 - \epsilon$ for any $b \in B \setminus A$, and $\mu(c) = 0$ for any $c \notin A \cup B$. Here ϵ is small enough for $\mu(B)$ to be larger than $\mu(A)$. The greedy algorithm will output A (verify!). However, the optimal solution is B. ∎

Remark 1.8: (last!) Due to the confusion in class, we have in fact managed to learn something new. As I was shown by some smart people in class, a stronger claim is in fact true:

Theorem 1.9: *Let $\mathcal{F}$ be a family of subsets of $\{1...n\}$ containing $\emptyset$. Then a greedy algorithm (as defined above) returns an optimal solution for any weight function iff $\mathcal{F}$ has the structure of a family of independent sets of a matroid.*

Note that the requirement that $\mathcal{F}$ is non-empty and hereditary has been replaced by a weaker one. ∎