1.

$$\begin{pmatrix} -1 & 3 & 2 \\ -2 & 4 & 5 \\ -2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1.5 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -1.5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1.5 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -1.5 \end{pmatrix}$$

2.

$$\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix}$$

3. (a) We know that ℓ_2 is a vector norm, so we just have to check it's a matrix norm as well. Now for every to square matrices A, B we have

$$\|AB\|_2^2 = \sum_{i,j} (AB)_{ij}^2 = \sum_{i,j} (\sum_k a_{ik} b_{kj})^2 \leq \sum_{i,j} (\sum_k a_{ik}^2) \cdot (\sum_k b_{kj}^2) = (\sum_{i,k} a_{ik}^2) \cdot (\sum_{j,k} b_{kj}^2) = \|A\|_2^2 \|B\|_2^2$$

(b) i. $|||A|||_{p_1 \to p_2} = \max_{x \neq 0} \frac{||Ax||_{p1}}{||x||_{p2}} \geq 0$ as $||,||_{p_1}$ is a norm. If $|||A|||_{p_1 \to p_2} = 0$ then $||Ax||_{p_1} = 0$ for every x, which means that $Ax = 0$ for every x as $||,||_{p_1}$ is a norm. But then $A = 0$.

ii. $|||cA|||_{p_1 \to p_2} = \max_{x \neq 0} \frac{||cAx||_{p1}}{||x||_{p2}} = \max_{x \neq 0} \frac{|c|||Ax||_{p1}}{||x||_{p2}} = |c| \max_{x \neq 0} \frac{c||Ax||_{p1}}{||x||_{p2}} = |c||||A|||_{p_1 \to p_2}$

iii. $|||A + B|||_{p_1 \to p_2} = \max_{x \neq 0} \frac{||(A+B)x||_{p1}}{||x||_{p2}} \leq \max_{x \neq 0} \frac{||Ax||_{p1} + ||Bx||_{p1}}{||x||_{p2}} \leq \max_{x \neq 0} \frac{||Ax||_{p1}}{||x||_{p2}} + \max_{x \neq 0} \frac{||Bx||_{p1}}{||x||_{p2}} = |||A|||_{p_1 \to p_2} + |||B|||_{p_1 \to p_2}$

4. (a) As A is symmetric then due to the spectral theorem A is orthogonally diagonisable, that is, there are U orthogonal and D diagonal such that $A = UDU^t$.

$$||A||_{2 \to 2} = \max_{x \neq 0} \frac{||Ax||_2}{||x||_2} = \max_{x \neq 0} \frac{||UDU^t x||_2}{||x||_2} = \max_{x \neq 0} \frac{||Dx||_2}{||x||_2} = \max_{x \neq 0} \sqrt{\frac{\sum_i (\lambda_i x_i)^2}{\sum_i x_i^2}} = \max_i |\lambda_i|$$

(b) Denote the i-th column of A by v_i. We have for any $x \neq 0$

$$\frac{||Ax||_p}{||x||_1} = \frac{|| \sum_i x_i v_i ||_p}{\sum_j |x_j|} \leq \sum_i \frac{|x_i| \|v_i\|_p}{\sum_j |x_j|} \leq \max_i \|v_i\|_p$$

Thus $|||A||||_{1 \to p} = max_i \|v_i\|_p$.

5. We want to find real x, y, z such that $xb + yp + z$ for b the bagrut mean and p the psychometric test result will predict in the best way the B.Sc. mean m. In other words, we want to find the minimum.

$$\min_{x,y,z} \left\| \begin{pmatrix} b_1 & p_1 & 1 \\ b_2 & p_2 & 1 \\ . & . & . \\ b_n & p_n & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} m_1 \\ m_2 \\ . \\ m_n \end{pmatrix} \right\|$$

and this can be solved by least-squares.