Algorithms - Solution 1

1. Pseudo-code for $\frac{2}{3}$-merge-sort:

   $\frac{2}{3} - Merge - Sort(Arr, p, q)$

   if $(p < q)$ then

   $\frac{2}{3} - Merge - Sort(Arr, p, p + \lfloor \frac{2(q-p)}{3} \rfloor)$
   $\frac{2}{3} - Merge - Sort(Arr, p + \lfloor \frac{2(q-p)}{3} \rfloor + 1, q)$
   $Merge(Arr, p, q)$

   where $Merge$ acts like the one from regular $Merge - Sort$, except setting the "middle" at $\lfloor \frac{2(q-p)}{3} \rfloor$ instead of $\lfloor \frac{q-p}{2} \rfloor$.
   Run-time analysis:
   First, lets prove that $T(n) = O(n \log n)$, i.e. $T(n) \leq c * n \log n$ for some $c > 0$ for all $n > n_0$ by induction. The recurrence equation of the function is $T(n) = T(\frac{2n}{3}) + T(\frac{n}{3}) + O(n)$, where $O(n)$ is the bound on $Merge$, since it reads every element of the array. The base step is $T(1) = 1$ (easy). The induction hypothesis is that for each $k < n$, $T(k) \leq c * k \log k$. Now, $T(n) = T(\frac{2n}{3}) + T(\frac{n}{3}) + O(n) \leq T(\frac{2n}{3}) + T(\frac{n}{3}) + c_1 * n$. Then, by induction hypothesis $T(n) \leq c * (\frac{2n}{3}) * \log(\frac{2n}{3}) + c * (\frac{n}{3}) * \log(\frac{n}{3}) + c_1 * n \leq \frac{2}{3}cn \log(n) + \frac{2}{3}cn \log(\frac{2}{3}) + \frac{1}{3}cn \log(n) + \frac{1}{3}cn \log(\frac{1}{3}) + c_1 * n = cn \log(n) + \frac{2}{3}cn \log(2) - \frac{2}{3}cn \log(3)) + \frac{1}{3}cn \log(1) - \frac{1}{3}cn \log(3)) + c_1 * n = cn \log(n) + \frac{2}{3}cn - cn \log(3) + c_1 * n$. We will choose c s.t. $-c_1 * n = \frac{2}{3}cn - cn \log(3)$, i.e. $c_1 = (\log(3) - \frac{2}{3})c$. With such choice $T(n) \leq c * n \log n$ and we proved the induction step.
   We can prove that $T(n) = \Omega(n \log n)$ in a similar way. Thus we get, $T(n) = \Theta(n \log n)$.

2. (a) Assume $d_1 = d_2$. We'll prove that $p_1(n) = \Theta(p_2(n))$, i.e. $\lim_{n \to \infty} \frac{p_1(n)}{p_2(n)} = c$. We can use L'Hospital rule to calculate this limit, which says that if both numerator and denominator of the quotient tend to infinity, the limit of the quotient is equal to the limit of derivative of the numerator divide by the derivative of denominator . We will derive d_1 times and get the limit $\lim_{n \to \infty} \frac{a_{d_1} + \frac{a_{d_1 - 1}}{n} + \ldots + \frac{a_0}{n^{d_1}}}{b_{d_1} + \frac{b_{d_1 - 1}}{n} + \ldots + \frac{b_0}{n^{d_1}}} = \lim_{n \to \infty} \frac{a_{d_1}}{b_{d_1}} = c$.

   Assume $p_1(n) = \Theta(p_2(n))$. Assume, by contradiction, that $d_1 \neq d_2$, w.l.o.g. $d_1 > d_2$. We know, that each polynomial is Θ of its highest degree term, i.e. $p_1 = \Theta(n^{d_1})$ and $p_2 = \Theta(n^{d_2})$. But, $d_1 > d_2$, so $n^{d_1} = \omega(n^{d_2})$. By transitivity, $p_1(n) = \omega(p_2(n))$, reaching the contradiction.

   (b) Assume $d_1 < d_2$. We'll prove that $p_1(n) = o(p_2(n))$, i.e. $\lim_{n \to \infty} \frac{p_1(n)}{p_2(n)} = 0$. We can use L'Hospital rule to calculate this limit, since both polynomials tends to infinity. We will derive d_1 times and get the limit $\lim_{n \to \infty} \frac{a_{d_1} + \frac{a_{d_1 - 1}}{n} + \ldots + \frac{a_0}{n^{d_1}}}{b_{d_1} n^{d_2 - d_1} + b_{d_1 - 1} n^{d_2 - d_1 - 1} + \ldots + \frac{b_0}{n^{d_1}}} = \lim_{n \to \infty} \frac{c}{\infty} = 0$.

Assume $p_1(n) = o(p_2(n))$. Assume, by contradiction, that $d_1 \geq d_2$. We proved in *a* that $d_1 \neq d_2$, so we need to check $d_1 > d_2$. We know, that each polynomial is Θ of its highest degree term, i.e. $p_1 = \Theta(n^{d_1})$ and $p_2 = \Theta(n^{d_2})$. But, $d_1 > d_2$, so $n^{d_1} = \omega(n^{d_2})$. By transitivity, $p_1(n) = \omega(p_2(n))$, reaching the contradiction.

3. The order of growth is: $\log^2(n), n^{1/3}, \log(n!), n^{\log\log n}, n!$.

   (a) Using L'Hospital rule:
   $\lim_{n\to\infty} \frac{\log^2(n)}{n^{1/3}} = \lim_{n\to\infty} \frac{2*\log(n)*\frac{1}{n}}{\frac{n^{-2/3}}{3}} = lim_{n\to\infty} \frac{6*\log(n)}{n^{1/3}} = \lim_{n\to\infty} \frac{\frac{6}{n}}{\frac{n^{-2/3}}{3}} = \lim_{n\to\infty} \frac{18}{n^{1/3}} = 0$

   (b) Using the fact that $\log(n!) = \Theta(n\log(n))$:
   $\lim_{n\to\infty} \frac{n^{1/3}}{n\log(n)} = \lim_{n\to\infty} \frac{1}{n^{2/3}\log(n)} = 0$

   (c) Notice that for each n: $n! \leq n^n$ and for each $n > 16$: $n^2 \leq n^{\log\log n}$. So, for each $n > 16$ we have: $\log(n!) \leq \log(n^n) = n * \log(n) \leq n^2 \leq n^{\log\log n}$

   (d) Apply log on both sides: $\log(n^{\log(\log(n))}) = \log(\log(n)) * \log(n) \leq \log(n) * \log(n) = \log^2(n) \leq \log(n!) \Rightarrow n^{\log(\log(n))} \leq n!$

4. $G = (V, E)$ is a tree, i.e. connected acyclic graph.

   (a) Assume we add an edge (v, u). Before the adding, because a graph is connected, we had a path from u to v, lets say $u, a_1, \ldots, a_k, v$, so now we have a path $u, a_1, \ldots, a_k, v, u$, which is a cycle.

   (b) Assume we remove an edge (v, u), and assume by contradiction that the graph remains connected, i.e. we still have a path from u to v, lets say $u, a_1, \ldots, a_k, v$, that means, that before removing we had a path $u, a_1, \ldots, a_k, v, u$, which is a cycle, reaching a contradiction.

   (c) We have at least one path between every pair of vertices, because a graph is connected. Assume, by contradiction, that we have two pathes between (v, u), lets say $p_1 = u, a_1, \ldots, a_k, v$ and $p_2 = u, b_1, \ldots, b_m, v$, that means that we have a path $p = u, a_1, \ldots, a_k, v, b_m, \ldots, b_1, u$, which is a cycle, reaching a contradiction.

5. The answer is $\log^*(\log n) = \omega(\log(\log^* n))$. This is because by definition $\log^* n$ counts the number of times we have to concatenate the log function with itself until we get a number smaller than 1. So you can think of $\log^*(\log n)$ as if we already applied log once and then we start counting. So $\log^* n = \log^*(\log n) + 1$. Or in other words $\log^* n = \Theta(\log^*(\log n))$. So now replace $\log^* n$ with the variable m. So we have on the one hand $\log^*(\log n) = \Theta(m)$ and on the other hand $(\log(\log^* n)) = \log m$. And then $\log^*(\log n) = \omega(\log(\log^* n))$.