Algorithms - Exercise 11

Due Thursday 20/1 24:00

1. Compute the LUP decomposition for

$$A = \begin{pmatrix} -1 & 3 & 2 \\ -2 & 4 & 5 \\ -2 & 3 & 4 \end{pmatrix}$$

   Write down the intermediate steps.

2. Let $M(n)$ be the time to multiply $n \times n$ matrices and let $S(n)$ denote the time required to square an $n \times n$ matrix. Show that multiplying and squaring matrices have essentially the same difficulty: $S(n) = \Theta(M(n))$.

3. Recall that a vector norm $||\cdot||$ should satisfy:

   (a) $\forall v \in \mathcal{R}^n$ $||v|| \geq 0$ and $||v|| = 0$ only when $v = 0$.

   (b) $\forall v \in \mathcal{R}^n$ and $\forall c \in \mathcal{R}$, $||cv|| = |c| \cdot ||v||$.

   (c) $\forall u, v \in \mathcal{R}^n$, $||u + v|| \leq ||u|| + ||v||$.

   A matrix norm should satisfy all of the above (where vectors are replaced by matrices) as well as the condition:

$$||AB|| \leq ||A|| \cdot ||B||$$

   Prove that the following two are matrix norms:

   (a) The ℓ_2 norm defined as: $|||A|||_2 = \left(\sum a_{ij}^2\right)^{\frac{1}{2}}$.

   (b) The operator norm defined as: $|||A|||_{p_1 \to p_2} = \max_x \frac{||Ax||_{p_1}}{||x||_{p_2}}$. Where $1 \leq p_1, p_2 \leq \infty$, and $||\cdot||_p$ is the vector norm ℓ_p (i.e. $||v||_p = (\sum_i v_i^p)^{\frac{1}{p}}$).

4. (a) Prove that $|||A|||_{2\to 2} = |\lambda|$, where λ is the eigenvalue of A with maximal absolute value.

   (b) Prove that $|||A|||_{1\to p}$ equals to the maximum of the vector norm ℓ_p of the columns of A.

5. Help Limor Livnat!
   Some people claim that there is a correspondence between certain tests (high school exams, psychometric test) and success in academic studies. But which one is more important? Suppose you are given a list of students together with each student's bagrut mean, psychometry result and B.Sc mean. Suggest an algorithm for computing the "sehem" in the best way possible.