Algorithms - Exercise 8

Due Wednesday 29/12 24:00

1. Prove that $det(V(x_0, \ldots, x_{n-1})) = \Pi_{0 \le j < k \le n-1}(x_k - x_j)$

   Where det is the determinant function and $V(x_0, \ldots, x_{n-1})$ is the Vandermonde matrix defined by $V(x_0, \ldots, x_{n-1})_{ij} = x_i^j$ $(0 \le i, j \le n-1)$.

   Hint: for every j, $0 \le j \le n-2$ multiply the j-th column by x_{n-1} and subtract it from $j+1$-th column. Argue that the matrix obtained has the same determinant as $V(x_0, \ldots, x_{n-1})$ and use induction.

2. (a) Let $p(x)$ be a polynomial with degree bound n, and let $x_0 \in \mathbb{R}$. Define $q(x)$ to be the polynomial of degree bounded by $n-1$ obtained by dividing $p(x)$ by the polynomial $x - x_0$. Namely, $p(x) = q(x)(x - x_0)$. Show how to compute in time $O(n)$ the coefficients of the polynomial q given x_0 and the coefficients of the polynomial p.

   (b) Show how to compute in time $O(n^2)$ a polynomial of degree bounded by n in coefficient representation from its point-value representation. that is, you are given $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$ (where $x_0, \ldots, x_{n-1}$ are distinct), and you have to compute the coefficients $a_0, \ldots, a_{n-1}$ of the unique polynomial $p(x)$ of degree less than n such that $p(x_i) = y_i$ for every $0 \le i \le n-1$.

   Hint: Use the interpolation polynomial as given by Lagrange's formula

   $$p(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \ne k}(x - x_j)}{\prod_{j \ne k}(x_k - x_j)}$$

   First compute $\Pi_{j=0}^{n-1}(x - x_j)$, and then divide by $x - x_k$ according to the formula.

3. Give the coefficient representation for the following polynomials given in point-value form:

   (a) $(1, 1), (-1, -1), (0, -1)$

   (b) $(1, 0), (i, 1), (-1, 0), (-i, -1)$

4. An n-th root of unity z is called a *primitive* root of unity if for every n-th root of unity a one has $a = z^k$ for some integer k.

   (a) Write down all the 8-th roots of unity and all the primitive 8-th roots of unity.

   (b) How many primitive n-th roots of unity are there if n is a prime number?

5. (a) Prove that $\sum_{k=0}^{n-1} \omega_n^{kj} \omega_n^{-kl} = n\delta_{jl}$

   (b) Show that $\frac{1}{n} V(\omega_n^0, \omega_n^{-1}, \ldots, \omega_n^{-(n-1)})$ is the inverse of $V(\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1})$