Algorithms - Exercise 7

Due Wednesday 15/12 24:00

1. Consider the following approximation algorithm for MAX-LIN-2:

   Input: A system of m linear equations over the field Z_2 with variables $x_1, \ldots, x_n$.

   Algorithm: In a loop until there are no more equations: Pick the equation with the smallest number of variables. Let's call these variables $y_1, \ldots, y_k$. Let $e_1, \ldots, e_\ell$ be all the equations in the system that include exactly the variables $y_1, \ldots, y_k$. Find an assignment to $y_1, \ldots, y_k$ that satisfies at least half of the equations $e_1, \ldots, e_\ell$ (show why such an assignment exist, and how it can be found efficiently). Now remove the equations $e_1, \ldots, e_\ell$ from the system, and remove all the appearances of variables from $y_1, \ldots, y_k$ by giving them the assignments that we found. Note that now you get a new system of linear equations with less equations and less variables.

   Prove that the algorithm is a 2-approximation. Analyze its running time as a function of m and n.

2. Let $f(X, Y) = \sum_{x \in X, y \in Y} f(x, y)$. Show

   (a) If $X, Y \subseteq V$ then $f(X, Y) = -f(Y, X)$

   (b) If $X \subseteq V$ then f(X,X)=0

   (c) If $X, Y, Z \subseteq V$ with $X \cap Y = \emptyset$ then

   $$f(X \cup Y, Z) = f(X, Z) + f(Y, Z) \quad \text{and} \quad f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$$

3. (a) Prove that flows in a network form a convex set. That is, show that for any two flows f_1, f_2 and $0 \leq t \leq 1$ the function $g : E \to \mathbb{R}$ given by

   $$g(e) = (1 - t)f_1(e) + tf_2(e) \quad \text{for } e \in E$$

   is a flow.

   (b) State the maximum-flow problem as a linear-programming problem.

4. Show that a maximum flow in a network $G = (V, E)$ can always be found by a sequence of at most $|E|$ augmenting paths.

5. Prove that if the capacity function c takes only integral values, then the maximum flow f produced by the Ford-Fulkerson method has the property that $|f|$ is integer-valued. Moreover, for all vertices u and v, the value of $f(u, v)$ is an integer.

6. A perfect matching in a graph is a matching in which every vertex is matched. Let $G = (L, R, E)$ be an undirected bipartite graph with $|L| = |R|$. For any $A \subseteq L \cup R$ we define $N(A)$ to be the set of neighbors of vertices from A. More formally,

$$N(A) = \{u \in L \cup R : \exists v \in A \text{ such that } (u, v) \in E\}$$

   Prove Hall's Matching Theorem using flow networks:

   There exists a prefect matching in G if and only if for every $A \subseteq L$, $|A| \leq |N(A)|$.