Algorithms - Exercise 3

Due Thursday 11/11 24:00

1. Consider the unit-time task scheduling problem dealt with in the class.
   (a) Show how to determine in $O(|A|)$ time whether or not a given set A of tasks is independent.
   (b) Show that the following algorithm for the problem works: Let all n time slots be initially empty, where time slot i is the unit-length slot of time that finishes at time i. We consider the jobs in order of monotonically decreasing penalty. When considering job j, if there exists a time slot at or before j's deadline d_j that is still empty, assign job j to the latest such slot, filling it. If there is no such slot, assign job j to the latest of the as yet unfilled slots.

2. (Reshut)
   (a) Let n, m be two natural numbers. Let S be any set of size n. Take I to be the family of all subsets of S not greater than m. Show that $U_{n,m} = (S, I)$ is a matroid. (Matroids of this type are called uniform matroids).
   (b) Show that if (S, I) is a matroid, then (S, I') is a matroid, where $I' = \{A' : S - A'$ contains some maximal $A \in I\}$

3. (a) Recall that a bipartite graph, $G = (L, R, E)$, is composed of two disjoint sets of vertices L and R and its edges can be only those that cross between L and R (i.e. there are no edges between the vertices of L and no edges between the vertices of R). More formally: $E \subseteq \{(v_1, v_2) : v_1 \in L \text{ and } v_2 \in R\}$

   Let $G = (L, R, E)$ be a bipartite graph. Prove that the following pair (S, I) is a matroid:
   - $S = L$
   - $I = \{A \subseteq L : \text{ there is a perfect matching between } A \text{ and a subset of } R\}$

   Recall that a matching in a bipartite graph is a subset of the edges in the graph such that each vertex in the graph is touched by at most one edge from the subset. A perfect matching is a matching that touches all the vertices in the graph.

   Hint: Note that the union of every two matchings can have three types of connected components: a simple cycle that includes the whole component, a simple path of odd length, and a simple path of even length.

   (b) Give an algorithm that solves the following matching problem. A matching agency publishes a list of n men, m women enter the agency, and each one of them gives a sublist of the n men that she wants to meet. Each woman also specifies the amount of money she will pay in the case she meets one of these men. Find a matching that maximizes the profit of the agency.