Simultaneous SDR Optimality via a Joint Matrix Decomposition

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***Abstract*—This work considers the joint source-channel problem of transmitting a Gaussian source over a two-user multiple-input multiple-output (MIMO) broadcast channel. We show the existence of non-trivial channels, where the optimal distortion pair (which for high signal-to-noise ratios equals the point-to-point distortions of the individual users) may be achieved. A condition for existence of a joint triangularization of the MIMO channels which shapes the ratio of the diagonals to a desired form is derived. Whenever possible, all diagonal elements but one are made equal. We then employ a hybrid digital-analog scheme to the source, where the digital part is sent over the equal subchannels and the analog refinement is sent over the remaining one.**

I. Introduction

The choice of modulation domain plays a major role in communication, both in deriving performance limits and in the design of practical schemes which decouple the signal processing task of channel equalization from coding. For example, the capacity of the Gaussian inter-symbol interference (ISI) channel is given by the water-filling solution, applied in the frequency domain; the same transformation also allows to use popular schemes such as Orthogonal Frequency-Division Multiplexing (OFDM) which employs the discrete Fourier transform. The singular value decomposition (SVD) plays a similar role for multiple-input multiple-output (MIMO) channels. Common to both cases is *diagonalization*: they yield parallel independent equivalent channels. Capacity, however, can be achieved for both the ISI and MIMO channels using non-orthogonal equivalent channels, by a receiver which performs *triangularization* of the channel[1] (rather than diagonalization) and then decision-feedback equalization or successive interference cancellation (SIC). This is done without performing any transformation at the encoder. It is therefore natural to ask, what can be achieved by allowing *both* an encoder transformation (in addition to the decoder one) and SIC.

One such direction, pursued by Jiang, Hager and Li [1], is the *generalized triangular decomposition* (GTD): a matrix A is decomposed as

$$A = UTV^{\dagger}\,, \tag{1}$$

where U and V are unitary matrices, $V^{\dagger}$ is the complex conjugate of V and T is upper-triangular.[2] It is shown that the transforming matrices U and V exist if and only if the diagonal elements of T obey a multiplicative majorization relation with the singular values of A. Since the product of these diagonal elements equals the product of the singular values of A, the decomposition performs *diagonal shaping*: it distributes the total gain between the diagonal elements in a desired way. An important special case is where it is desired to have balanced gains, i.e., the diagonal elements of T should be equal (allowing for equal-rate codebooks) [2]. In that case, named the *geometric mean decomposition* (GMD), the majorization condition holds for any A.

We take a different path, in which we strive to jointly shape the diagonals of two matrices, for the purpose of multi-terminal communication. Since with this approach the choice of basis depends upon more than one communication link, we call it *network modulation*. We jointly triangularize two matrices A_1 and A_2 as

$$A_i = U_i T_i V^{\dagger}\,, \quad i = 1, 2, \tag{2}$$

where U_1, U_2 and V are unitary and T_1 and T_2 are upper-triangular. Having the same matrix V on one of the sides of the decomposition corresponds to applying the same transformation, and is thus suitable to two links originating (or terminating) at the same node. It turns out that in different network applications, it is important to shape the vector of *ratios* between the diagonals. We show that the relevant condition for achievability of a ratio vector is a multiplicative majorization relation with the generalized singular values [3] of the pair (A_1, A_2). As in the single-user case, this condition is always satisfied if we are interested in a constant ratio vector. In [4] this was used to introduce a transmission scheme for broadcasting digital data over two MIMO links.

In this paper we prove the sufficient and necessary condition for the achievability of a general ratio vector, and use it for the problem of transmission of an *analog* source over two MIMO links, or, in information-theoretic terms, joint source-channel coding (JSCC) of a source over a broadcast (BC) channel.

Transmission over a BC channel is indeed one of the main applications of JSCC, as in such a scenario it may be greatly superior to source-channel separation. In a classical

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[1] Outside the high signal-to-noise ratio regime, "near triangularization" is performed as an optimal balance between residual interference and noise.

[2] An upper-triangular matrix of dimensions $m \times n$ is defined as one having zero entries beneath its main diagonal.

example, a white Gaussian source needs to be transmitted over a single-input single-output (SISO) additive white Gaussian noise (AWGN) channel with unknown signal-to-noise ratio (SNR), with one channel use per source sample, under mean-squared error (MSE) distortion. For our purposes we consider the case that the SNR may take one of two values, which we may think of as corresponding to a setup with two users, a "strong" one (having a high SNR), and a "weak" one (lower SNR). While analog transmission is known to achieve the optimal performance for any channel SNR, the separation-based scheme (concatenation of successive-refinment and broadcast codes) yields a tradeoff, where if we wish to be optimal for the weak user, then both users have the same distortion, while optimality for the strong user means that the distortion for the other user is trivial (equals the source variance).

In this work we replace the SISO links by MIMO ones, so that there is a mismatch in degrees of freedom. In particular, the problem subsumes the better known problem of transmission over a colored and/or bandwidth-mismatched Gaussian BC [5], [6]; however, none of the schemes proposed are strictly optimal. We present a simple outer bound on the achievable distortions region of the MIMO BC channel, as well as propose an inner bound which achieves it for high SNR, in certain cases. For the case of two transmit antennas, non-asymptotic optimality is shown for certain cases. This applies to some cases of color and bandwidth mismatch, although not to the white bandwidth (BW) expansion one.

II. Background: MIMO Channels and Unitary Triangularization

In this section we show how the single-user MIMO capacity may be achieved using multiple SISO codebooks with SIC over an equivalent channel obtained by unitary triangularization of the form (1). The exposition follows the universal matrix decomposition (UCD) [2], which is in turn based upon the derivation of the MMSE version of V-BLAST. Later in the paper we take the triangularization to be one which is simultaneously good for two users. We will further consider in the sequel hybrid analog-digital (HAD) transmission rather than fully digital; these issues are suppressed for now. We assume throughout the paper perfect channel knowledge everywhere.

We consider a point-to-point (complex) MIMO channel:

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{z}\,, \tag{3}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the channel input and output vectors of dimensions $N_t \times 1$ and $N_r \times 1$, respectively; H is the channel matrix of dimensions $N_r \times N_t$ and $\boldsymbol{z}$ is the additive Gaussian noise vector of dimensions $N_r \times 1$. Without loss of generality, we assume that the noise elements are mutually-independent, identically-distributed and circularly-symmetric with unit variance. The capacity of this channel is given by

$$C(H,P) = \max_{C_{\boldsymbol{x}}} I(H, C_{\boldsymbol{x}}) \tag{4}$$

where the maximization is over all channel input covariance matrices $C_{\boldsymbol{x}}$, subject to a power constraint $\mathrm{trace}\,(C_{\boldsymbol{x}}) \leq P$, and $I(H,C_{\boldsymbol{x}}) \triangleq \log\det\left(I + HC_{\boldsymbol{x}}H^\dagger\right)$ is the maximal mutual information (MI) attainable using covariance matrix $C_{\boldsymbol{x}}$.

When coding over a domain different than the channel input domain (e.g., time or space), one may start with a virtual input vector $\tilde{\boldsymbol{x}}$, related to the physical input by the linear transformation: $\boldsymbol{x} = C_{\boldsymbol{x}}^{1/2} V \tilde{\boldsymbol{x}}$. [3] We form the vector $\tilde{\boldsymbol{x}}$ in turn by taking one symbol from each of N_t parallel codebooks, of equal power $1/N_t$, and the unitary matrix V is the linear precoder which performs the basis transformation. Recalling the GTD (1), one may suggest to choose V by applying a triangularization to $F \triangleq HC_{\boldsymbol{x}}^{1/2}$.

After the receiver applies the transformation $U^\dagger$, it is left with an equivalent triangular channel T, over which it may decode the codebooks using SIC. Unfortunately, while this "conserves" the determinant of $HC_{\boldsymbol{x}}H^\dagger$, it fails to do so when the identity matrix is added as in the mutual information $I(H,C_{\boldsymbol{x}})$ (4). Thus, this is optimal in the high SNR limit only, and an MMSE variation is needed in general, as follows.

We start by applying unitary triangularization to an augmented matrix:

$$\begin{bmatrix} F \\ I \end{bmatrix} \triangleq G = UTV^\dagger, \tag{5}$$

where the identity matrix I has dimensions $N_t \times N_t$. At the receiver we compute: $\tilde{\boldsymbol{y}} = W\boldsymbol{y}$, where W consists of the first N_t rows of U. This results in an equivalent channel:

$$\tilde{\boldsymbol{y}} = W(FV\tilde{X} + \boldsymbol{z}) = WW^\dagger T\tilde{\boldsymbol{x}} + W\boldsymbol{z} \triangleq \tilde{T}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{z}}.$$

Note that since W is not unitary, the statistics of $\tilde{\boldsymbol{z}}$ differ from those of $\boldsymbol{z}$; denote the covariance matrix of the equivalent noise by $C_{\boldsymbol{z}} = WW^\dagger$. Finally, SIC is performed, i.e., the codebooks are decoded from last to first, using:

$$y'_j = \tilde{y}_j - \sum_{l=j+1}^{N_t} \tilde{T}_{j,l}\hat{\tilde{x}}_l\ ,$$

where $\hat{\tilde{x}}_l$ is the decoded symbol from the l-th codebook. Assuming correct decoding of "past" symbols, i.e. $\hat{\tilde{x}}_l = \tilde{x}_l$ for all $l > j$, the scalar channel for decoding of the j-th codebook has signal-to-intereference and noise ratio (SINR)

$$S_j = \mathrm{Var}\left(\tilde{\mathrm{x}}_\mathrm{j}\middle|\tilde{\mathbf{y}}, \tilde{\mathrm{x}}_{\mathrm{j+1}}^{\mathrm{N_t}}\right) = \frac{\tilde{T}_{j,j}^2}{[C_{\boldsymbol{z}}]_{j,j} + \sum_{l=1}^{j}\tilde{T}_{j,l}^2}\,. \tag{6}$$

The following shows optimality of the scheme.

Proposition 1: Suppose that for some channel H and input covariance matrix $C_{\boldsymbol{x}}$, the augmented channel matrix can be triangularized as in (5). Then the decomposition satisfies:

$$\sum_{j=1}^{N_t} \log T_{j,j}^2 = I(H, C_{\boldsymbol{x}})\,, \tag{7}$$

and the SINRs S_j (6) of the transmission scheme above satisfy:

$$1 + S_j = T_{j,j}^2\,, \quad \forall j = 1,\ldots,N_t\,. \tag{8}$$

For a proof see [2, Lemma III.3 and Corollary III.4]. This proposition completes the recipe for a digital transmission scheme which achieves $I(H,C_{\boldsymbol{x}})$: for a given input covariance

[3] The square root of a Hermitian positive-definite matrix A, denoted by $A^{1/2}$, is defined as the matrix satisfying: $A = A^{1/2}\left(A^{1/2}\right)^\dagger$.

matrix $C_{\boldsymbol{x}}$, choose the individual codebook rates to approach

$$R_j = \log T_{j,j}^2 \,. \tag{9}$$

By (8), the successive decoding procedure will succeed with arbitrarily low error probability for these rates. By (7), the sum of the codebook rates equals the MI over the channels. Taking the covariance matrix $C_{\boldsymbol{x}}$ maximizing (4), achieves capacity.

III. Joint Triangularization with Shaped Diagonal Ratio

In this section we prove the necessary and sufficient condition for the existence of the joint triangularization (2). Throughout the section, we make the following assumption on the dimensions and ranks of the matrices.

Definition 1 (Proper dimensions): An $m \times n$ matrix A is said to have proper dimensions if it is full-rank and $m \geq n$. Note that augmented matrices of the form (5) satisfy this condition for any channel matrices. For stating the existence condition, we need the following definitions.

Definition 2 (Generalized singular values [3]): For any (ordered) matrix pair (A_1, A_2), the generalized singular values (GSVs) are the positive solutions a of the equation

$$\det\left\{A_1^\dagger A_1 - a^2 A_2^\dagger A_2\right\} = 0\,.$$

Let the GSV vector $\boldsymbol{\mu}(A_1, A_2)$ be the vector of all GSVs (with their algebraic multiplicity), ordered non-increasingly.[4]

For matrices of proper dimensions, $\boldsymbol{\mu}$ is of length n.

Remark 1: When A_1 and A_2 are square and non-singular, $\boldsymbol{\mu}(A_1, A_2)$ consists of the singular values of $A_1 A_2^{-1}$.

Definition 3 (Diagonal ratios vector): Let T_1 and T_2 be two upper-triangular matrices of proper dimensions $m_1 \times n$ and $m_2 \times n$, respectively, with non-negative diagonal elements. The diagonal ratios vector $\boldsymbol{r}(T_1, T_2)$ is the n-length vector which contains all ratios $T_{1;j,j}/T_{2;j,j}$, ordered non-increasingly.

Definition 4 (Multiplicative majorization): Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two n-dimensional vectors satisfying $\prod_{j=1}^n |x_j| = \prod_{j=1}^n |y_j|$. Then we say that $\boldsymbol{x}$ majorizes $\boldsymbol{y}$ ($\boldsymbol{x} \succeq \boldsymbol{y}$) if for any $1 \leq k < n$,

$$\prod_{j=1}^k |x_j| \geq \prod_{j=1}^k |y_j| \,.$$

We are now ready to prove the main result of this section.

Theorem 1: Let A_1 and A_2 be two matrices of proper dimensions $m_1 \times n$ and $m_2 \times n$, respectively. Then the joint unitary triangularization of (2) exists iff $\boldsymbol{\mu}(A_1, A_2) \succeq \boldsymbol{r}(T_1, T_2)$.

Proof idea: The direct follows the same lines as the proof in [4] for the constant ratio between diagonals case, by using the GTD with diagonal $\boldsymbol{r}(T_1, T_2)$ instead of the GMD. Denote the $n \times n$ upper sub-matrices of T_1 and T_2 by $[T_1]$ and $[T_2]$, respectively. The converse is based upon the fact that $\boldsymbol{\mu}(A_1, A_2) = \boldsymbol{\mu}(T_1, T_2) = \boldsymbol{\mu}([T_1], [T_2])$.

Remark 2: By the unitarity of U and V, the products of $\boldsymbol{\mu}$ and $\boldsymbol{r}$ are equal. Thus, the majorization relations mean that the diagonal ratios are always "less spread" than the GSVs.

[4] We define a GSV to be infinite, if the corresponding GSV of the matrices in reverse order is zero. If the number of finite and infinite solutions is smaller than n, this suggests that the column rank can be reduced without changing the problem; we shall assume the problem is in its reduced form.

Remark 3 (Relation to GSVD): The GSVD [3] can be stated in a triangular form (2), with diagonals ratio $\boldsymbol{r}(T_1, T_2) = \boldsymbol{\mu}(A_1, A_2)$. Thus, the GSVD is a limiting case with maximal ratio spread.

Remark 4 (Relation to GTD): Taking in the joint decomposition $H_2 = I$ yields the GTD of H_1 [1]; further, the GSV become the singular values vector of H_1. The existence condition, in turn, reduces to the Weyl condition (see e.g. [1]). In this sense, the condition in Theorem 1 may be seen as a generalized Weyl condition for joint triangularization.

Remark 5 (Relation to the generalized Schur decomposition): This decomposition, also called the QZ-decomposition [7], is a special case of (2) with $U_1 = U_2$. It can be shown that the diagonal ratio vector induced by this decomposition is unique, i.e., requiring that the unitary matrices are the same on both sides prohibits shaping of the diagonal ratio.

IV. HDA Transmission for Source Multicasting

In the joint source-channel problem of interest, an i.i.d. circularly-symmetric Gaussian source S needs to be reproduced at two destinations, over a MIMO BC channel, i.e. the channel to each destination is given by (3), where the input $\boldsymbol{x}$ is common to both. We measure the quality of the reproductions $\hat{S}_i$ using the MSE distortion measure. Thus, we wish to maximize the tradeoff between the signal-to-distortion ratios (SDRs), defined as

$$\text{SDR}_i \triangleq \frac{\text{Var}\,(\text{S})}{\text{Var}\left(\hat{\text{S}}_\text{i} - \text{S}\right)}\,, \quad i = 1, 2\,. \tag{10}$$

The achievable SDR region $\mathcal{S}(H_1, H_2)$ is defined as the closure of all pairs which can be achieved by some encoding-decoding scheme. A simple outer bound on this region follows.

Proposition 2: $\mathcal{S}(H_1, H_2) \subseteq \bar{\mathcal{S}}(H_1, H_2)$, where the bounding region $\bar{\mathcal{S}}(H_1, H_2)$ is given by:

$$\bigcup_{C_{\boldsymbol{x}}:\text{trace}(C_{\boldsymbol{x}}) \leq P} \left\{(\text{SDR}_1, \text{SDR}_2) : \log(\text{SDR}_i) \leq I(H_i, C_{\boldsymbol{x}})\right\}.$$

The proof follows that of the classical source-channel converse [8], taking into account that both users share the same channel input. Our main result states that this is indeed achievable for some channels.

Theorem 2: Let $\boldsymbol{\mu}$ be the GSV vector of the augmented channel matrices (5) with input covariance matrix $C_{\boldsymbol{x}}$. Then SDR-pairs satisfying $\log \text{SDR}_i \leq I(H_i, C_{\boldsymbol{x}})$ are achievable, if

$$\prod_{j=1}^{N_t} \boldsymbol{\mu}_j \leq 1 \leq \prod_{j=1}^{N_t - 1} \boldsymbol{\mu}_j \,. \tag{11}$$

Proof: It follows by Theorem 1 that there exists a joint unitary triangularization with diagonal ratios vector which is all one except for the last element. The diagonal of T_i can thus be made to satisfy $T_{1;j,j} = T_{2;j,j} \triangleq t_j$ for $j = 1, \ldots, N_t - 1$. If we were to send digital data over the MIMO-BC channel using this particular triangularization, then by (7) we could send over these $N_t - 1$ channels a rate of:

$$R_{\text{digital}} \triangleq \sum_{j=2}^{N_t} R_j = \sum_{j=2}^{N_t} \log t_j^2.$$

This does not change if we replace, in the transmission scheme, $\tilde{x}_1$ by a different signal of the same variance P/N_t. Furthermore, regardless of the signal $\tilde{x}_1$, if the codebooks of subchannels $2, \ldots, N_t - 1$ are correctly decoded then, by Proposition 1, receiver i obtains an equivalent channel for $j = 1$ with an SNR of $\text{SNR}_{\text{analog},i} = (T_{i;1,1})^2 - 1$. At this stage we have turned the MIMO BC channel into the combination of a digital channel of rate R_{digital} and a SISO BC channel of signal-to-noise ratios $\text{SNR}_{\text{analog},i}$. Using an optimal HDA scheme with $N_t - 1$ digital layers and one analog provides [5]: $\log \text{SDR}_i = I(H_1, C_{\boldsymbol{x}})$, which completes the proof. ■

Theorem 2 does not imply that $\bar{\mathcal{S}}(H_1, H_2)$ is fully achievable. However, if the channel matrices are of proper dimensions, then in the limit of high SNR (as the choice $C_{\boldsymbol{x}} = I$ becomes optimal), the region of Theorem 2 coincides with $\bar{\mathcal{S}}(H_1, H_2)$. This is also the case for $N_t \leq 2$ (for any SNR). as is indicated by the following Lemma.

Lemma 1: Let H_1, H_2 be two matrices of proper dimensions, with $n = 2$ columns, and denote their corresponding augmented matrices (5) by G_1, G_2, resp., for some Hermitian matrix $C_{\boldsymbol{x}} \geq 0$. Then if one of the elements of $\boldsymbol{\mu}(H_1, H_2)$ is at least one and the other is at most one, then so are the elements of $\boldsymbol{\mu}(G_1, G_2)$.

Corollary 1: Let H_1, H_2 be channel matrices with $N_t = 2$. If $\boldsymbol{\mu}(H_1, H_2)$ is mixed, then the bounding region $\bar{\mathcal{S}}(H_1, H_2)$ of Proposition 2 is achievable.

V. BW Expansion over Two Parallel Channels

Consider the two-input two-output case:[5]

$$H_i = \begin{bmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{bmatrix}, \; i = 1, 2\,. \tag{12}$$

The bounding SDR region of Proposition 2 now becomes:

$$\bigcup_{0 \leq \gamma \leq 1} \Big\{ (\text{SDR}_1, \text{SDR}_2) : \\ \text{SDR}_i \leq \big(1 + |\alpha_i|^2 \gamma P\big)\Big(1 + |\beta_i|^2 (1-\gamma) P\Big) \Big\}. \tag{13}$$

Here γP is the portion of P sent over the first band.

We can point out a few special cases where points on the surface of this region are achievable by known strategies.

1) No BW expansion: analog transmission. If one of the bands has zero capacity, e.g., $\beta_i = 0$, (13) reduces to $\text{SDR}_i \leq 1 + |\alpha_i|^2 P$ – achievable by analog transmission.
2) Equal SDRs: digital transmission. A point on the boundary which satisfies $\text{SDR}_1 = \text{SDR}_2$ may be achieved by quantizing the source and then using a digital common-message code for the BC channel.
3) One equal band: HDA transmission. If for one of the bands the gains are equal, e.g., $|\beta_1| = |\beta_2| = \beta$, we can quantize the source and use a point-to-point digital code over that band. The quantization SDR may approach $\text{SDR}_{\text{digital}} = 1 + \beta^2 \gamma P$. The quantization error may be sent over the other band in an analog manner, yielding an additional gain of $1 + |\alpha_i|^2 (1-\gamma) P$ at each decoder, achieving the bound (13).

[5]Being diagonal, this channel may be obtained from a single-input single-output Gaussian inter-symbol interference channel which has a two-step frequency response, by applying the discrete Fourier transform.

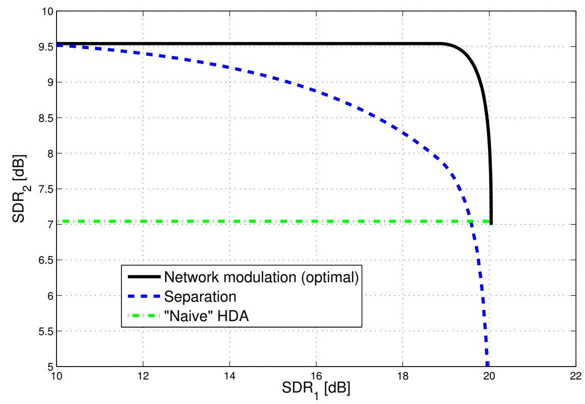

Fig. 1: Performance for $\alpha_1 = 1$, $\beta_1 = 10$,$\alpha_2 = \beta_2 = 2$,$P = 1$.

Using network modulation, we can extend the HDA transmission (case 3 above), by transforming a diagonal channel where none of the gains is equal between users, to an equivalent triangular channel where for one of the bands the gain is equal. This can be done under the condition (11), which specializes to (allowing to swap roles between matrices):

$$|\alpha_1|^2 \geq |\alpha_2|^2 \text{ and } |\beta_1|^2 \leq |\beta_2|^2 \tag{14}$$

or vice versa. This is an "anti-degradedness" condition: no user can have better SNR on both bands. It is not known whether this condition is necessary, but at least for the case where both channels are white ($\alpha_i = \beta_i$), it was shown in [6] that simultaneous optimality is *not* possible.

Figure 1 shows a numerical evaluation of performance for some gain values. It can be appreciated that the optimal performance imposes almost no tradeoff between users; indeed in the limit of high SNR, both can have their optimal point-to-point performance. For comparison, we show the performance of a separation-based scheme, where a successive-refinement source code is transmitted over a digital broadcast channel code, as well as that of a "naïve" HDA scheme, where transmission is digital over one band and analog over the other.

References

[1] Y. Jiang, W. Hager, and J. Li. The generalized triangular decompostion. *Mathematics of Computation*, vol. 77, no. 262:1037–1056, Apr. 2008.
[2] Y. Jiang, W. Hager, and J. Li. Uniform channel decomposition for MIMO communications. *IEEE Trans. SP*, vol. 53:4283–4294, Nov. 2005.
[3] C. F. Van Loan. Generalizing the singular value decomposition. *SIAM J. Numer.*, Anal. 13:76–83, 1976.
[4] A. Khina, Y. Kochman, and U. Erez. Decomposing the MIMO broadcast channel. In *48th Annual Allerton Conference*, Sept. 29–Oct. 1, 2010.
[5] U. Mittal and N. Phamdo. Hybrid digital-analog (HDA) joint source-channel codes for broadcasting and robust communications. *IEEE Trans. IT*, 48:1082–1103, May 2002.
[6] Z. Reznick, M. Feder, and R. Zamir. Distortion bounds for broadcasting with bandwidth expansion. *IEEE Trans. IT*, 52:3778–3788, Aug. 2006.
[7] G. H. Golub and C. F. Van Loan. *Matrix Computations, 3rd ed.* Johns Hopkins University Press, Baltimore, 1996.
[8] C. E. Shannon. Coding theorems for a discrete source with a fidelity criterion. In *Institute of Radio Engineers, International Convention Record, Vol. 7*, pages 142–163, 1959.