Graph Decomposition Lemmas and their Role in Metric Embedding Methods

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Abstract. We present basic graph decomposition lemmas and show how they apply as key ingredients in the probabilistic embedding theorem stating that every n point metric space probabilistically embeds in ultrametrics with distortion $O(\log n)$ and in the proof of a similar bound for the spreading metrics paradigm in undirected graphs. This provides a unified framework for these metric methods which have numerous algorithmic applications.

1 Introduction

This paper presents new graph decomposition lemmas which lie at the heart of two metric-based techniques that have been the basis for many results in approximation and online algorithms in recent years: probabilistic metric embedding in ultrametrics [2] and the spreading metrics paradigm [10]. We thus provide a unified framework for these techniques.

Our main graph decomposition lemma is similar in spirit to the well-known graph decompositions of Garg, Vazirani and Yannakakis [15] and Seymour [19].

Such a decomposition finds a low diameter cluster in the graph such that the weight of the cut created is small respect to the weight of the cluster. These type of decompositions have been extremely useful in the theory of approximation algorithms, e.g. for problems such as minimum multi-cut, balanced separators, and feedback arc-set. Our new decomposition is essentially based on the decomposition of [15] performed in a careful manner so as to achieve a more refined bound on the ratio between the cut and the cluster's weight. The graph decomposition lemma is presented in Section 2.

We also give a new probabilistic decomposition lemma. Probabilistic partitions, introduced in [2], are partitions of the graph into low diameter clusters with small probability of cutting any edge. Such partitions have been used in the context of probabilistic embeddings and are shown over time to have a yet more significant role in the theory of metric embeddings. Here, we present a variation on this idea. Instead of partitioning the entire graph we focus on constructing one low diameter cluster, as in the previous lemma case. Roughly, the decomposition has the property that the probability an edge is cut is small with respect

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to the probability that it is included in the cluster. The lemma is essentially based on the decomposition of [2] preformed in a careful manner so as to achieve a more refined bound on the ratio between these probabilities. The probabilistic decomposition lemma is presented in Section 4.

Probabilistically embedding metric spaces in ultrametrics (and hierarchically well-separated trees) was introduced in [2]. The original metric space is embedded in a randomly chosen (dominating) ultrametric so that the distortion is kept low in expectation. Such an embedding is extremely useful for algorithmic application since it reduces the problem of obtaining an approximation algorithm for the original space to obtaining an algorithm for ultrametrics, a task which is often considerably easier. It has implications for embeddings in low dimensional normed spaces as well [7]. In [2] we have introduced probabilistic partitions and proved that such a partition gives a probabilistic embedding in ultrametrics. In [3] we have proved that every metric space on n points can be probabilistically embedded in ultrametrics with distortion $O(\log n \log\log n)$, based on the technique of [19]. Recently,the bound was improved to $O(\log n)$ [13] by providing an improved probabilistic partition, which is based on techniques from [9, 12]. This bound is tight [2]. The theorem has been used in many applications, including metric labelling [17], bulk-at-buy network design [1], group Steiner tree [14], min-sum clustering [5], metrical task systems [4, 11], etc. (see the survey [16] for a more comprehensive summary of applications.) The approximation algorithm for some of these problems can be made deterministic via methods of [8] by using the dual problem to probabilistic embedding, which we call distributional embedding: given a distribution over the pairs in the metric space construct a single ultrametric as to approximate the expected distance. It follows from duality that the best solutions for these two problems have the same distortion bound. Moreover, it is shown in [3, 8] that a solution to the dual problem can be used to construct an efficient probabilistic embedding. In [13] such a dual solution is obtain by applying derandomization techniques to their probabilistic algorithm. Here, we use our graph decomposition lemma to give a simple direct construction of a distributional embedding in ultrametrics with $O(\log n)$ distortion. This implies simpler approximation algorithms in the applications. This is done in Section 3.

We next turn to the probabilistic embedding version, and show how our probabilistic decomposition lemma, together with techniques from [3], yield a simple direct efficient construction of probabilistic embedding in ultrametrics with $O(\log n)$ distortion. This appears in Section 4.

A related notion of embedding is that of multi-embeddings introduced in [6]. We note that our graph decomposition lemma applies to obtain improved results for multi-embeddings in ultrametrics as well.

Spreading metrics were introduced in [10]. They consider minimization problems defined in a graph G with capacities and assume the existence of a spreading metric which provides a lower bound on the cost of a solution for the problem as well as some guaranty which relates the distances to the cost on smaller parts of the problem. In the context of this paper we restrict the graph G to be undirected. This method has been applied to several problems, including linear arrangement, balanced partitions, etc. An approximation ratio of $O(\log n \log\log n)$ was given for these problems [10], based on the technique of [19]. Most of these were later improved in [18] to $O(\log n)$. However, their method was not general and excluded, in particular, the problem of embedding graphs in d-dimensional meshes, where we would like to minimize the average distance in the embedding. In this paper we improve the generic spreading metric bound to $O(\log n)$. We note that the spreading metrics paradigm applies to some other problems in directed graphs, including directed feedback arc-set. The directed case remains a challenging open problem. We discuss the application of our graph decomposition lemma to the spreading metrics paradigm in Section 5.

2 Graph Decompositions

2.1 Decomposition Lemma

Let $G = (V, E)$ be an undirected graph with two weight functions $c, x : E \to \mathbb{R}^+$. We interpret $x(e)$ to be the length of the edge e, and the distance between pairs of vertices u, v in the graph, denote $d(u, v)$, is determined by the length of the shortest path between them. Given a subgraph $H = (V_H, E_H)$ of G, let d_H denote the distance in H, let $\Delta(H)$ denote the diameter of H, and $\Delta = \Delta(G)$. We define the volume of H, $\phi(H) = \sum_{e \in E_H} x(e)c(e)$.

Given a subset $S \subseteq V$, $G(S)$ denotes the subgraph of G induced by S. Given partition $(S, \bar{S})$ let $\Gamma(S) = \{(u, v) \in E; u \in S, v \in \bar{S}\}$ and $\mathrm{cut}(S) = \sum_{e \in \Gamma(S)} c(e)$.

For a vertex v and $r \geq 0$, the ball at radius r around v is defined as $B(v, r) = \{u \in V | d(u, v) \leq r\}$. Let $S = B(v, r)$. Define

$$\bar{\phi}(S) = \bar{\phi}(v, r) = \sum_{e=(u,w); u,w \in S} x(e)c(e) + \sum_{e=(u,w) \in \Gamma(S)} c(e)(r - d(v, u)).$$

Given a subgraph H, we can similarly define $\bar{\phi}_H$ with respect to the subgraph H. Define the spherical-volume of H,

$$\phi^*(H) = \max_{v \in H} \bar{\phi}_H(v, \Delta(H)/4).$$

Lemma 1. *Given a graph G, there exist a partition $(S, \bar{S})$ of G, where S is a ball and*

$$\mathrm{cut}(S) \leq \frac{8 \ln\left(\phi^*(G)/\phi^*(G(S))\right)}{\Delta(G)} \cdot \bar{\phi}(S).$$

Proof. Let v be a node that minimizes the ratio $\bar{\phi}(v, \Delta/4)/\bar{\phi}(v, \Delta/8)$. We will choose $S = B(v, r)$ for some $r \in [\Delta/8, \Delta/4]$. We apply a standard argument, similar to that of [15]. First observe that $\frac{d\bar{\phi}(v,r))}{dr} = \mathrm{cut}(B(v,r))$. Therefore there exist r such that

$$\mathrm{cut}(B(v,r))/\bar{\phi}(v,r) \leq \left(\int_{r=\Delta/8}^{\Delta/4} \frac{d\bar{\phi}(v,r)}{\bar{\phi}(v,r)}\right) / \left(\frac{\Delta(G)}{8}\right)$$
$$= \frac{\ln\left(\bar{\phi}(v,\Delta/4)/\bar{\phi}(v,\Delta/8)\right)}{\Delta(G)/8}.$$

Now, let u be the node that maximizes $\bar{\phi}_S(u, \Delta(G(S))/4)$. Since $\Delta(G(S)) \leq \Delta/2$ we have that $\phi^*(G(S)) \leq \bar{\phi}(u, \Delta/8)$. Recall that $\phi^*(G) \geq \bar{\phi}(u, \Delta/4)$. By the choice of v we have

$$\mathrm{cut}(S) \leq \frac{8\ln\left(\bar{\phi}(u,\Delta/4)/\bar{\phi}(u,\Delta/8)\right)}{\Delta(G)} \cdot \bar{\phi}(v,r)$$
$$\leq \frac{8\ln\left(\phi^*(G)/\phi^*(G(S))\right)}{\Delta(G)} \cdot \bar{\phi}(S).$$

□

2.2 Recursive Application

The decomposition comes useful where it is applied recursively. We show in Sections 3 and 5 that our applications posses a cost function α over subgraphs $\hat{G}$ of G which is nonnegative, 0 on singletons, and obeys the recursion:

$$\alpha(\hat{G}) \leq \alpha(\hat{G}(S)) + \alpha(\hat{G}(\bar{S})) + \Delta(\hat{G}) \cdot \mathrm{cut}(S). \tag{1}$$

Lemma 2. *The function defined by (1) obeys $\alpha(G) \leq O(\log(\phi/\phi_0)) \cdot \phi(G)$ where $\phi = \phi(G)$ and ϕ_0 is the minimum value of $\phi(\hat{G})$ on non-singleton subgraphs $\hat{G}$.*

Proof. We prove by induction on the size of subgraphs $\hat{G}$ of G that $\alpha(\hat{G}) \leq 8\ln(\phi^*(\hat{G})/\phi_0) \cdot \phi(\hat{G})$. Let $\hat{G}$ be a subgraph of G. Using lemma 1 for $\hat{G}$ we obtain $S = B(v,r)$ so that

$$\begin{aligned}
\alpha(\hat{G}) &\leq \alpha(\hat{G}(S)) + \alpha(\hat{G}(\bar{S})) + \Delta(\hat{G}) \cdot \mathrm{cut}(S) \\
&\leq 8\ln\left(\phi^*(\hat{G}(S))/\phi_0\right) \cdot \phi(\hat{G}(S)) + 8\ln\left(\phi^*(\hat{G}(\bar{S}))/\phi_0\right) \cdot \phi(\hat{G}(\bar{S})) \\
&\quad\quad + \quad 8\ln\left(\phi^*(\hat{G})/\phi^*(\hat{G}(S))\right) \cdot \bar{\phi}(v,r) \\
&\leq 8\ln\left(\phi^*(\hat{G})/\phi_0\right) \cdot \bar{\phi}(v,r) + 8\ln\left(\phi^*(\hat{G})/\phi_0\right) \cdot \phi(\hat{G}(\bar{S})) \\
&\leq 8\ln\left(\phi^*(\hat{G})/\phi_0\right) \cdot \phi(\hat{G}).
\end{aligned}$$

The result follows since $\phi^*(\hat{G}) \leq \phi(\hat{G})$. □

To obtain a bound depending only on n, we again apply a standard argument from [15]: modify the process slightly by associating a volume $\phi(G)/n$ with nodes. This ensures that $\phi_0 \geq \phi(G)/n$.

Lemma 3. *The function defined by (1) using the modified procedure obeys $\alpha(G) \leq O(\log n) \cdot \phi(G)$.*

3 Embedding in Ultrametrics

Definition 1. *An* ultrametric *(HST*[1]*) is a metric space whose elements are the leaves of a rooted tree T. Each vertex $u \in T$ is associated with a label $\Delta(u) \geq 0$ such that $\Delta(u) = 0$ iff u is a leaf of T. It is required that if a u is a child of a v then $\Delta(u) \leq \Delta(v)$. The distance between two leaves $x, y \in T$ is defined as $\Delta(\mathrm{lca}(x, y))$, where $\mathrm{lca}(x, y)$ is the least common ancestor of x and y in T.*

We study the set of metric spaces defined over an n-point set V. Given such a metric space M, the distance between u and v in V is denoted $d_M(u, v)$. It is useful to identify the metric space with a complete weighted graph $G = (V, E)$, where edge weights $x(e)$ are distances in M.

The following definitions are from [2].

Definition 2. *A metric space N* dominates *another metric space N if for every $u, v \in V$, $d_N(u, v) \geq d_M(u, v)$.*

Definition 3. *A metric space M α-probabilistically embeds in a class of metric spaces $\mathcal{N}$ if every $N \in \mathcal{N}$ dominates M and there exists a distribution $\mathcal{D}$ over $\mathcal{N}$ such that for every $u, v \in V$:*

$$\mathrm{E}[d_N(u, v)] \leq \alpha d_M(u, v).$$

The following notion serves as a dual to probabilistic embedding [3, 8].

Definition 4. *A metric space M α-distributionally embeds in a class of metric spaces $\mathcal{N}$ if every $N \in \mathcal{N}$ dominates M and given any weight function $c : V^2 \rightarrow \mathbb{R}^+$, there exists $N \in \mathcal{N}$ satisfying:*

$$\sum_{u,v \in V^2} c(u, v) d_N(u, v) \leq \alpha \sum_{u,v \in V^2} c(u, v) d_M(u, v).$$

It follows from duality considerations (see [3, 8]) that a metric space α-distributionally embeds in $\mathcal{N}$ iff it α-probabilistically embeds in $\mathcal{N}$.

Theorem 1. *Any metric space on n points $O(\log n)$-distributionally embeds in an ultrametric.*

Proof. We build an ultrametric recursively as follows: use the decomposition described in Section 2 to obtain a partition of the graph $(S, \bar{S})$. Run the algorithm on each part recursively, obtaining ultrametrics (HSTs) T_S and $T_{\bar{S}}$ respectively. An ultrametric (HST) T is constructed by creating a root r labelled Δ with two children at which we root the trees T_S and $T_{\bar{S}}$. Define $\mathrm{cost}(T) = \sum_{(u,v) \in E} c(u, v) d_T(u, v)$. Then

$$cost(T) = \mathrm{cost}(T_S) + \mathrm{cost}(T_{\bar{S}}) + \Delta \cdot \mathrm{cut}(S).$$

The theorem now follows from Lemma 3. □

[1] A k-HST [2] is defined similarly while requiring that $\Delta(u) \leq \Delta(v)/k$. Any ultrametric k-embeds [3] and $O(k/\log k)$-probabilistically embeds [5] in a k-HST.

4 Probabilistic Embedding

Theorem 1 implies the basic result on probabilistic embedding of arbitrary metrics in ultrametrics. Moreover, it is shown in [3, 8] how to efficiently construct the probabilistic embedding. In this section we combine ideas from our new graph decomposition together with ideas from [2, 3] to obtain the following probabilistic decomposition lemma, which is then applied to give a simple and efficient direct construction of a probabilistic embedding in ultrametrics.

4.1 Probabilistic Decomposition Lemma

Let $R = \Delta/C$, where $C = 128$. Define $\lambda(x) = |B(x, 16R)|/|B(x, R)|$. Let v be the vertex that minimizes $\lambda(v)$. Choose a radius r in the interval $[2R, 4R]$ according to the distribution[2] $p(r) = (\frac{\lambda(v)^2}{1-\lambda(v)^{-2}})\frac{\ln\lambda(v)}{R}\lambda(v)^{-\frac{r}{R}}$. Now define a partition $(S, \bar{S})$ where $S = B(v, r)$.

The distribution above is chosen to satisfy the following relation:

Lemma 4.

$$\Pr[(x,y) \in \Gamma(S)] \le \left[(1 - \Pr[x, y \in \bar{S}]) \cdot \ln\lambda(x) + \lambda(v)^{-1}\right] \cdot \frac{d(x,y)}{R}.$$

Proof. Assume w.l.o.g. that y is closer to v than x. We have:

$$\begin{aligned}\Pr[(x,y) \in \Gamma(S)] &= \int_{d(v,y)}^{d(v,x)} p(r)dr \\ &= (\tfrac{\lambda(v)^2}{1-\lambda(v)^{-2}})\lambda(v)^{-\frac{d(v,y)}{R}}(1 - \lambda(v)^{-\frac{d(v,x)-d(v,y)}{R}}) \\ &\le (\tfrac{\lambda(v)^2}{1-\lambda(v)^{-2}})\lambda(v)^{-\frac{d(v,y)}{R}} \cdot \frac{d(x,y)}{R}\ln\lambda(v), \end{aligned} \tag{2}$$

$$\begin{aligned}1 - \Pr[x, y \in \bar{S}] &= \int_{d(v,y)}^{4R} p(r)dr \\ &= (\tfrac{\lambda(v)^2}{1-\lambda(v)^{-2}})(\lambda(v)^{-\frac{d(v,y)}{R}} - \lambda(v)^{-4}). \end{aligned} \tag{3}$$

Therefore we have:

$$\begin{aligned}&\Pr[(x,y) \in \Gamma(S)] - (1 - \Pr[x, y \in \bar{S}]) \cdot \frac{d(x,y)}{R}\ln\lambda(x) \\ &\le (\tfrac{\lambda(v)^2}{1-\lambda(v)^{-2}})\lambda(v)^{-4} \cdot \ln\lambda(v) \cdot \frac{d(x,y)}{R} \le \lambda(v)^{-1} \cdot \frac{d(x,y)}{R},\end{aligned}$$

where we have used the fact that $\lambda(x) \ge \lambda(v)$, and the last inequality holds for any value $\lambda(v) \ge 1$. This completes the proof of the lemma. □

[2] This is indeed a probability distribution. Note that we may assume $\lambda(v) > 1$ otherwise the partition is trivial.

4.2 The Probabilistic Embedding Theorem

Theorem 2. *Any metric space on n points $O(\log n)$-probabilistically embeds in an ultrametric.*

Proof. We build an ultrametric recursively as follows: use the probabilistic decomposition described above to obtain a partition of the graph $(S, \bar{S})$. Run the algorithm on each part recursively, obtaining ultrametrics (HSTs) T_S and $T_{\bar{S}}$ respectively. An ultrametric (HST) T is constructed by creating a root r labelled Δ with two children at which we root the trees T_S and $T_{\bar{S}}$.

The analysis shows that for every $(x, y) \in E$:

$$\frac{\mathrm{E}[d_T(x,y)]}{C \cdot d(x,y)} \leq \ln |B(x, 16R)| + \ln |B(x, 8R)|.$$

The proof is by induction on the construction of the HST. We have

$$\begin{aligned}\mathrm{E}[d_T(x,y)] = \Pr[(x,y) \in \Gamma(S)]\Delta \\ + \Pr[x, y \in S] \cdot \mathrm{E}[d_{T_S}(x,y)] + \Pr[x, y \in \bar{S}] \cdot \mathrm{E}[d_{T_{\bar{S}}}(x,y)]. \quad (4)\end{aligned}$$

We now apply the induction hypothesis. We assume that $\Pr[(x,y) \in \Gamma(S)] \neq 0$, otherwise the result follows immediately from the induction hypothesis. This implies that $B(v, 4R)$ includes x or y. We may also assume $d(x,y) \leq R/2$, otherwise the result is immediate. It follows that $d(v,x) < 5R$ and so that $B(v,R) \subseteq B(x, 8R)$.

$$\begin{aligned}\frac{\mathrm{E}[d_{T_{\bar{S}}}(x,y)]}{C \cdot d(x,y)} &\leq \ln |B(x,16R) - S| + \ln |B(x, 8R) - S| \\ &\leq \ln |B(x, 16R)| + \ln(|B(x, 8R)| - |B(v, R)|), \quad (5)\end{aligned}$$

Since $\Delta(S) \leq 8R$ we have:

$$\frac{\mathrm{E}[d_{T_S}(x,y)]}{C \cdot d(x,y)} \leq \ln |B(x,R)| + \ln |B(x, R/2)|,$$

and since $B(v, 2R)$ does not include both x and y, we may assume that $d(v,x) \geq 2R - d(x,y) \geq 3/2 \cdot R$. Thus, we have that $B(x, R/2) \subseteq B(x, 8R) - B(v, R)$, which gives:

$$\frac{\mathrm{E}[d_{T_S}(x,y)]}{C \cdot d(x,y)} \leq \ln |B(x,R)| + \ln(|B(x, 8R)| - |B(v,R)|). \quad (6)$$

We apply Lemma 4. Recall that $\lambda(v)^{-1} = |B(v,R)|/|B(v,16R)|$. Since $d(v,x) < 5R$ we have that $B(x,8R) \subseteq B(v,16R)$, hence $\lambda(v)^{-1} \leq |B(v,R)|/|B(x,8R)|$. Thus the lemma implies:

$$\begin{aligned}\Pr[(x,y) \in \Gamma(S)] \leq [(1 - \Pr[x, y \in \bar{S}]) \cdot \ln(|B(x, 16R)|/|B(x,R)|) \\ + |B(v,R)|/|B(x,8R)|] \cdot \frac{d(x,y)}{R}. \quad (7)\end{aligned}$$

We get the inductive claim by plugging the above inequalities in (4):

$$\begin{aligned}\frac{\mathrm{E}[d_T(x,y)]}{Cd(x,y)} &\leq (1-\Pr[x,y\in\bar{S}])\left[\ln\frac{|B(x,16R)|}{|B(x,R)|}+\ln|B(x,R)|\right]\\ &\quad+\Pr[x,y\in\bar{S}]\cdot\ln|B(x,16R)|\\ &\quad+\left[\ln(|B(x,8R)|-|B(v,R)|)+\frac{|B(v,R)|}{|B(x,8R)|}\right]\\ &\leq \ln|B(x,16R)|+\ln|B(x,8R)|.\end{aligned}$$

□

5 Spreading Metrics

The spreading metric paradigm [10] applies to minimization problems on undirected graphs $G=(V,E)$ with edge capacities $c(e)\geq 1$ for $e\in E$. Associated is an auxiliary graph H and a scaler function on subgraphs of H. A decomposition tree T is a tree with nodes corresponding to non-overlapping subsets of V, forming a recursive partition of V. For a node t of T denote by V_t the subset at t. Associated are the subgraphs G_t, H_t induced by V_t. Let F_t be the set of edges that connect vertices that belong to different children of t, and $c(F_t)=\sum_{e\in F_t}c(e)$. The cost of T is $\mathrm{cost}(T)=\sum_{t\in T}\mathrm{scaler}(H_t)\cdot c(F_t)$.

Definition 5. *A* spreading metric *a function $x: E\to \mathbb{R}^+$ satisfying: $\sum x(e)c(e)$ is a lower bound on the optimal cost, and for any $U\subseteq V$, and H_U the subgraph of H induced by U, $\Delta(U)\geq \mathrm{scaler}(H_U)$.*

Theorem 3. *There exist an $O(\log n)$ approximation for problems in the spreading metrics paradigm.*

Proof. The algorithm is defined recursively: use the decomposition described in Section 2 to obtain a partition of the graph $(S,\bar{S})$. Run the algorithm on each part recursively. A tree T is constructed by creating a root r with two children at which we root the trees T_S and $T_{\bar{S}}$. The cost of the tree is given by

$$\begin{aligned}\mathrm{cost}(T) &= \mathrm{cost}(T_S)+\mathrm{cost}(T_{\bar{S}})+\mathrm{scaler}(H)\cdot\mathrm{cut}(S)\\ &\leq \mathrm{cost}(T_S)+\mathrm{cost}(T_{\bar{S}})+\Delta\cdot\mathrm{cut}(S).\end{aligned}$$

The result now follows from Lemma 3. □

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