Approximating Directed Multicuts

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Abstract

The seminal paper of Leighton and Rao (1988) and numerous papers that followed it resolved the question of relating multicommodity flow values to cut capacities in undirected networks, created the divide-and-conquer method for designing approximation algorithms, and generated novel tools for utilizing linear programming relaxations. Yet, despite persistent research efforts, these achievements could not be duplicated in the case of directed networks, excluding a few special cases that are "symmetric" and therefore similar to the undirected case. This paper is an attempt to remedy this situation. We consider the problem of finding a minimum multicut in a directed multicommodity flow network, and give the first non-trivial upper bounds on the max flow-to-min multicut ratio. Our results are algorithmic, demonstrating non-trivial approximation guarantees.

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1 Introduction

A *network* is a graph $G = (V, E)$, directed or undirected, with positive edge capacities $c : E \rightarrow \mathbb{R}^+$, together with a list of source-sink pairs of vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$, sometimes called *commodities*. Computational problems in networks often deal with two basic concepts: *cuts* and *flows*. Informally, a cut is a set of edges whose removal disconnects some or all of the source-sink pairs. The most interesting statistic about a cut is its capacity, the total capacity of all the edges in the cut. A flow is a mathematical formulation of how fluids, or electrical currents, or similar physical entities, can be transported from the sources to the sinks without violating the capacity constraints. Here, the most interesting statistic is the value of the flow, the total amount of flow that reaches the sinks. The duality between flow value and cut capacity in networks is a fundamental phenomenon in combinatorial optimization, with numerous important applications, both in theory and in practice. The celebrated single commodity max flow-min cut theorem of Ford and Fulkerson [7] would appear on any short list of the most influential results in the field. The theorem has been generalized to some special cases of multicommodity instances (see [14] for a comprehensive discussion). However, it was observed early on that such a strong duality relation does not hold for arbitrary multicommodity flow instances.

In their ground-breaking work, Leighton and Rao [15] prove that for uniform multicommodity flow instances the sparsest cut-to-maximum concurrent flow ratio in undirected networks is at most logarithmic in the number of vertices.[1] They exhibit several applications of this result, mostly in the design and analysis of approximation algorithms for NP-hard optimization problems. Their paper inspired a significant research effort in the past decade. The results of this effort include the emergence of the divide-and-conquer method in approximation algorithms (see [22]), applications of their region-growing technique to other problems [2, 11, 8, 21], and the development of alternative proofs for their basic result and its generalizations [16, 1, 5]. In particular, Garg, Vazirani, and Yannakakis [8] give an elegant analysis of the region-growing technique, and use it to derive asymptotically tight $O(\log k)$ bounds on the minimum multicut-to-maximum flow ratio in k-commodity undirected networks (improving weaker bounds implicit in Klein, Agrawal, Ravi, and Rao [11]).[2]

Most of the previous research on approximation algorithms for flow/cut problems in directed networks exploits some sort of "symmetry" property that renders the problems similar to the undirected case; for example, the commodities occur in symmetric pairs $(s_i, t_i),\ (t_i, s_i)$ [15, 21, 13, 6, 4, 5, 18]. In particular, for such symmetric instances, Even, Naor, Schieber, and Sudan [6], improving upon a result of Klein, Plotkin, Rao, and Tardos [13], gave an $O((\log k) \log \log k)$ bound, and they gave efficient algorithms to find a "symmetric multicut" whose capacity is within the same factor of the optimum. (A symmetric multicut means a set of arcs whose removal disconnects either s_i from t_i or t_i from s_i, for every symmetric pair of commodities.) These papers use region-growing techniques, though the bounds that are proved are usually weaker than those that can be proven in the undirected case.

Unfortunately, there is *no* relation between a (directed) multicut and a symmetric multicut. For example, consider a directed graph on two nodes p, q with two arcs (p, q) and (q, p) having capacities 1 and 1000, respectively. There are two commodities $(s_1, t_1) = (p, q)$ and $(s_2, t_2) = (q, p)$. The unique multicut has capacity 1001, whereas there is a symmetric multicut of capacity one. Another way to see the contrast is to compare the integrality ratios of the linear programming relaxations: it is $O((\log k) \log \log k)$ for symmetric multicuts [6] but a construction due to Saks et al. (discussed below) shows that it is k for directed multicuts [20]. To put it bleakly, the literature cited in the previous paragraph has almost no relevance for (asymmetric) directed multicuts.

Despite persistent research effort, very little is known about flow/cut duality in (asymmetric) *directed* networks or about approximating the cut problems that arise in this context. Also, there is growing evidence that problems

[1] The sparsity of a cut is the ratio between the cut capacity and the number of source-sink pairs that are disconnected. A concurrent flow delivers the same amount of flow of each commodity.

[2] A multicut is a set of edges whose removal disconnects all commodities, and the result of [8] relates the capacity of a multicut to the maximum total flow value for all commodities. This is perhaps the most straightforward generalization of flow/cut duality in the single-commodity case.

related to flow/cut duality in directed networks are much harder than their undirected counterparts. The simplest example is that the two-commodity multicut problem in directed networks is NP-hard [9], whereas the undirected version can be solved in polynomial time [10]. Furthermore, known methods like region-growing, low-distortion embeddings, and spreading metrics seem to "miss the point" for arbitrary instances in directed networks. This situation is not unique to flow/cut duality problems. For example, network design problems (also related to cuts) suffer from the same sorry state. Thus, the study of directed networks is a major challenge and a significant part of the agenda of expanding our knowledge on the approximability of NP-optimization problems.

In this paper, we consider the problem of finding a minimum multicut in networks (without any symmetry assumptions), "network" without "undirected" meaning "directed network" from now on, and provide the first non-trivial upper bounds relating cut capacities to flows. For a network G, we denote by $C(G)$ the capacity of the minimum multicut, and by $F(G)$ the value of the maximum multiflow. (For undirected networks G', we denote the corresponding quantities by $C'(G')$ and $F'(G')$.)

We prove four related theorems. Each of these theorems gives a bound on $C(G)$ in terms of $F(G)$ and other parameters of the network G; moreover, each proof gives an efficient algorithm for finding a multicut whose capacity is at most the bound on $C(G)$. The bounds given by the four theorems are mutually incomparable in the sense that for each of the four bounds, there exist networks where that bound is better than the other three bounds.

Theorem 1. There is a polynomial-time algorithm that takes a network G satisfying $c(e) \geq 1$ for all e and finds a multicut M satisfying $c(M) \leq 108 \cdot F(G)^3$.

We prove that without the "$c(e) \geq 1$ for all e" condition, no result of the form "$C(G) \leq g(F(G))$ for all G" is possible. (For undirected graphs, Yannakakis [24] shows, via a variant of the region-growing procedure of [8], that $C(G) = O(F(G) \log F(G))$, if all capacities are at least 1.)

Theorem 2. There is a polynomial-time algorithm that takes a k-commodity network G satisfying $c(e) \geq 1$ for all e and finds a multicut M satisfying
$$c(M) \leq 39 \ln(k+1) F(G)^2.$$

(Again, the "$c(e) \geq 1$ for all e" condition is necessary.)

Theorem 3. There is a polynomial-time algorithm that takes an n-node, k-commodity network G and finds a multicut M satisfying
$$c(M) \leq (45\sqrt{n \ln(k+1)})F(G) \leq (45\sqrt{2}\sqrt{n \ln n})F(G)$$

We give a better approximation guarantee for some instances in planar digraphs.

Theorem 4. For every Δ, there are a c and a polynomial-time algorithm that takes any n-node, k-commodity network G with $k \geq 2$, having uniform capacities, in which the total degree of every vertex is at most Δ and whose underlying undirected graph is planar, and finds a multicut M in G satisfying $c(M) \leq (c\sqrt{\lg k})n^{1/4}F(G)$.

(Tardos and Vazirani [23] use the methods of Klein, Plotkin, and Rao [12] to prove a constant ratio for undirected planar networks.)

Theorem 1 is our basic result. The other three theorems are based on it, and derived using techniques such as region growing (Theorem 2), a trade-off via LP rounding (Theorem 3), and a trade-off via the planar separator theorem (Theorem 4).

In recent work, Saks, Samorodnitsky, and Zosin [20] construct a family of k-commodity networks, for all k and $\epsilon > 0$, where the minimum multicut-to-maximum k-commodity flow ratio is no less than $k - \epsilon$, in contrast with the $O(\log k)$ upper bound in the undirected case. (An upper bound of k is a trivial consequence of the Ford and Fulkerson theorem.) We note that in their graphs, $|V|$ is exponential in k, so an upper bound of $O(\log |V|)$, for example, is still possible. In fact, the networks in [20] have special structure. Each is obtained by adding $2k$ distinct new vertices $s_1, t_1, ..., s_k, t_k$ to an *undirected* graph H, together with arcs from the s_i's to some vertices in H and from some vertices in H to the t_i's, replacing each undirected edge by a pair of antiparallel arcs, and assigning positive capacities to the *vertices*. Each terminal gets infinite capacity. We show in Section 4 that any network G of such simple form with $C(G) \geq (k/2)F(G)$ must, like the example of [20], have a number of nodes which is exponential in k. Indeed, the same result holds if capacities are instead assigned to arcs, provided that the arcs incident from the sources and to the sinks have infinite capacity.

The best inapproximability result known for directed multicut is that the problem is MAX SNP-hard. This is also the strongest hardness result known for the undirected case [3].

2 Preliminaries

A *network* G is a directed graph (V, E), without parallel arcs or self-loops, with an assignment of positive capacities to the arcs $c : E \to \mathbb{R}^+$, together with a positive integer k and a set of k distinct ordered pairs (s_i, t_i) of vertices, $s_i \neq t_i$ for all i. Let $T = \{s_1, t_1, s_2, t_2, ..., s_k, t_k\}$ be the set of *terminals*. A *multicut* M in G is a subset $M \subseteq E$ such that for no $i \in \{1, 2, ..., k\}$ is there an $s_i \to t_i$ path in the digraph $(V, E - M)$. (All paths are simple in this paper.) The *capacity* of a multicut M is $c(M) = \sum_{e \in M} c(e)$. DIRECTED MULTICUT is the problem of finding a minimum capacity multicut in a specified network G. Let us denote the minimum capacity of a multicut in G by $C = C(G)$. (When we work with undirected networks, the underlying graph G' is undirected and the minimum capacity of a multicut is denoted $C' = C'(G')$.)

The problem of finding a minimum capacity multicut in G is precisely this integer program: Find $x(e)$ for all $e \in E$, $x(e)$ integral, $x(e) \geq 0$ for all e, so as to minimize $\sum_{e \in E} c(e)x(e)$, such that for every $i = 1, 2, 3, ..., k$, and for every $s_i \to t_i$ path P in G, $\sum_{e \in P} x(e) \geq 1$. An optimal solution will have $x(e) \leq 1$ for all e.

Dropping the "$x(e)$ integral" condition gives a linear programming relaxation of DIRECTED MULTICUT: Find a nonnegative real length $x(e)$ for each arc such that for each i, the distance from s_i to t_i, relative to these lengths, is at least 1, so as to minimize $\sum c(e)x(e)$. Its linear programming dual is well known to be equivalent to MULTICOMMODITY FLOW, which is this problem: Given a network G, find a sequence $(f_1, f_2, ..., f_k)$ such that f_i is a single-source flow (of "commodity i") in G from source s_i to sink t_i, such that $(f_1, f_2, ..., f_k)$ satisfies $\sum_{1 \leq i \leq k} f_i(e) \leq c(e)$ for all e, and in which the sum over i of the value of f_i is maximized. Let $F = F(G)$ denote the optimal value of the multicommodity flow in G. It is easy to see that $C(G) \geq F(G)$ for all G. (In an undirected network, each f_i is a union of undirected $s_i - t_i$ paths, and the optimal flow value is denoted $F' = F'(G')$.) Since MULTICOMMODITY FLOW can be phrased as a polynomially-large linear program, it can be solved in polynomial time.

We are interested in how much larger $C(G)$ can be than $F(G)$ in an arbitrary network. Can $C(G)$ be bounded as a function of $F(G)$ for all G (regardless of the number of vertices and commodities)? In other words, is there a function $g : \mathbb{R} \to \mathbb{R}$ such that for all G, regardless of the number of nodes and commodities, $C(G) \leq g(F(G))$?

We will (easily) see below that if the capacities can be arbitrarily small, then the answer is no. However, if we insist only that $c(e) \geq 1$ for all e, then it is a nontrivial fact that $F(G) \leq 1$ implies $C(G) \leq 1$.

It is important to realize that DIRECTED MULTICUT is not a generalization of UNDIRECTED MULTICUT obtained by replacing each undirected edge by a pair of antiparallel arcs and by replacing $\{s_i, t_i\}$ by $(s_i, t_i), (t_i, s_i)$. For example, consider a four-node undirected tree with root r and leaves l_1, l_2, l_3. Suppose we define three commodities, one for each pair of distinct leaves, and make all capacities one. Since we must remove two edges to kill all $s_i - t_i$ paths, the minimum capacity of an undirected multicut is 2. However, if we now replace each edge by two arcs and define six commodities, one for each ordered pair of leaves, we see that it suffices to remove all arcs into the root, or three in total, not twice 2.

3 Algorithms and Bounds for Multicut

3.1 Multicut is Bounded By a Function of Flow

In this section we prove Theorem 1, that $C(G) \leq 108F(G)^3$, provided that $c(e) \geq 1$ for all e. But first we prove that such a result is not possible without the "$c(e) \geq 1$ for all e" assumption.

Garg, Vazirani, and Yannakakis show [8] that there exists $c > 0$ such that for all sufficiently large n, there is an n-node, undirected, unit-capacities network G'_n (on an expander), having $C'(G'_n)/F'(G'_n) \geq c \lg n$. Create a (directed) network G_n from G'_n by replacing each edge by a pair of antiparallel, unit-capacity arcs. We have $F(G_n) \leq F'(2G'_n) = 2F'(G'_n)$ (because any flow in G_n is feasible in $2G'_n$, which is G'_n with its capacities doubled) and $C(G_n) \geq C'(G'_n)$ (because if M is a minimum multicut in G_n, then $M' = \{\{u,v\} | (u,v) \in M$ or $(v,u) \in M\}$ is a multicut in G'_n and $|M'| \leq |M|$). Hence $C(G_n)/F(G_n) \geq C'(G'_n)/(2F'(G'_n)) \geq (c/2) \lg n$.

Now suppose that $C(G) \leq g(F(G))$ for all directed networks G. Choose a large enough n and set $\lambda = F(G_n)$. Let $H_n = G_n/\lambda$ (i.e., scale all capacities down by λ). We have

$$\frac{c}{2} \lg n = \frac{1}{\lambda}(\frac{c}{2} \lg n) F(G_n) \leq \frac{1}{\lambda} C(G_n) = C(H_n) \leq g(F(H_n)) = g(\frac{1}{\lambda} F(G_n)) = g(1),$$

which is a contradiction.

Now we prove Theorem 1.

Proof. We give a polynomial-time algorithm to construct a multicut of capacity at most $108F^3$ in a network G on digraph (V, E) satisfying $c(e) \geq 1$ for all e, where $F = F(G)$. First, find a nonnegative, rational length function x satisfying $\sum_e c(e)x(e) = F$ and $\sum_{e \in P} x(e) \geq 1$ for all $s_i \to t_i$ paths P, for all i. Define $f = \sum_e x(e) \leq \sum_e c(e)x(e) = F$.

For a technical reason, we need $x(e) \leq 1/6$ for all e. Replace any arc e with $x(e) > 1/6$ by a path of $\lceil 6x(e) \rceil$ new arcs of length at most $1/6$ each, whose lengths add to $x(e)$, all of whose capacities are $c(e)$.

We need some more definitions. For any $M \subseteq E$, define $c(M) = \sum_{e \in M} c(e)$. Let $E' \subseteq E$. Given any vertex s and real ρ, let $B_{E'}(s, \rho) = \{u \in V |$ there is an $s \to u$ path in (V, E') of length at most $\rho\}$. Define $Bdry_{E'}(s, \rho)$ $= \{(a, b) \in E' | a \in B_{E'}(s, \rho), b \notin B_{E'}(s, \rho)\}$.

For our purposes, *the prefix* of path $P =< u_0, u_1, u_2, ..., u_z >$ (whose length may exceed 1) is the path $P' =< u_0, u_1, u_2, ..., u_i >$ where i is minimal such that the length of P' (relative to x) is at least $1/6$, and *the suffix* of path $P =< u_0, u_1, u_2, ..., u_z >$ is the path $P' =< u_i, u_{i+1}, ..., u_z >$ where i is maximal such that the length of P' is at least $1/6$.

Here is the algorithm. The embedded comments are needed for the analysis.

/* Let $count(e) = 0$ for all $e \in E$. */

Let $E' = E$.

As long as there is a pair (s_i, t_i) such that some $s_i \to t_i$ path exists in $G' = (V, E')$, repeat:

1. Choose any such i.
   /* Find a shortest $s_i \to t_i$ path P_i in G' with respect to x. */
2. Among those ρ in $(1/3, 2/3)$, find a $\rho = \rho^*$ which minimizes $c(Bdry_{E'}(s_i, \rho))$.
   /* Let $B_i = B_{E'}(s_i, \rho^*)$. */
   /* Increment $count(e)$ for all e in the prefix of P_i. */
3. Remove from E' all arcs in $Bdry_{E'}(s_i, \rho^*)$.

Output $M = E - E'$.

End.

Obviously this process terminates and provides a multicut. We claim that the capacity of the multicut is at most $108f^2F \leq 108F^3$.

We need the following lemma, which is implicit in [8]. See also [22, p.204].

Lemma 5. Let $G = (V, E)$ be a digraph and let $s \in V$. Let $x : E \to \mathbb{R}^+$ be a length function, $c : E \to \mathbb{R}^+$ be a positive capacity function, and $E' \subseteq E$. Then there is a $\rho \in (1/3, 2/3)$ such that $c(Bdry_{E'}(s, \rho)) \leq 3F'$, where $F' = \sum_{e \in E'} c(e)x(e) \leq F$.

The lemma implies that in a given iteration we cut arcs of capacity at most $3F$.

Call the process of incrementing $count(e)$ *charging* e. In each iteration, we charge a set of arcs of total length at least $1/6$, all endpoints of which are in $B_{E'}(s_i, 1/3) \subseteq B_{E'}(s_i, \rho^*) = B_i$, because each arc has length at most $1/6$ and because $\rho > 1/3$. Since the total capacity added to $E - E'$ in an iteration is at most $3F$,

$$c(E - E') \leq 18F \sum_{e \in E} x(e) count(e)$$

is an invariant. We prove next that $count(e)$ never exceeds $6f$, and hence

$$c(E - E') \leq (18F)(6f) \sum_{e \in E} x(e) = 108f^2F.$$

Choose any e in the original G and relabel the commodities so that we charge e in the iterations for commodities $1, 2, ..., b$, in that order (and no others); these need not be consecutive iterations, of course. We claim, for $i = 1, 2, ..., b$, that:

1) All the vertices on the suffix of P_i are not in B_i.

2) All the vertices on the suffix of P_i are in $B_1 \cap B_2 \cap B_3 \cap \cdots \cap B_{i-1}$.

Now 1) is trivial, because we chose a ρ^* which is less than $2/3$, and no arc's length exceeds $1/6$; hence the endpoints of the suffix are not in B_i.

Proving 2) is not much harder. Since the iteration for commodity i charges e, P_i must contain e. Where $e = (u, v)$, v (and u) must be in $P_1, P_2, ..., P_b$. Consider now the portion Q of P_i extending from v to the end of P_i (Q contains the suffix of P_i). Choose $l < i$ and suppose, for a contradiction, that some vertex of Q is not in B_l. Since $v \in B_l$ (in the iteration for commodity l, all endpoints of arcs we charge are in B_l) and $t_i \notin B_l$, P_i contains some arc (w, x) with $w \in B_l, x \notin B_l$. But in the prior iteration for commodity l, we removed *all* arcs leaving ball B_l from E'! So this cannot happen, and the proof of the claim is complete.

We conclude that if $l < l'$, then the suffix of P_l is disjoint from the suffix of $P_{l'}$, because each vertex of the suffix of P_l is not in B_l and each vertex of the suffix of $P_{l'}$ is. This means that the sum of the lengths of arcs in G is at least $(1/6)b$, and hence $(1/6)b \leq f$, or $b \leq 6f$. ■

3.2 The Region-Growing Technique

Recall that the digraph is denoted $G = (V, E)$, each arc e has a positive capacity $c(e)$, and there are k commodities, each specified by a source-sink pair (s_i, t_i). Let each arc e have a nonnegative length $x(e)$, where x is a feasible solution to the linear program. Let $d_x(v, w)$ denote the shortest-path distance from vertex v to vertex w with respect to arc lengths x.

For a vertex set $S \subseteq V$, $c_E(S, V-S)$ denotes the sum of the capacities of arcs (v,w) in E in the cut, i.e., $c_E(S, V-S) = \sum_{(v,w)\in E, v\in S, w\notin S} c(v,w)$. Let $vol_E(S)$ denote the sum of $x(e)c(e)$ over all arcs $e \in E$ that have at least one end vertex (either tail or head) in S.

Recall that $F(G)$ denotes the optimal value of the linear program

$$\min\{\sum_e c(e)x(e) \quad : \quad d_x(s_i,t_i) \geq 1 \ (i=1,\ldots,k); \ x \geq 0\}$$

and that $vol_E(V) = F(G)$ if the length function x is optimal for the LP.

The next lemma extends Lemma 4.1 (on region growing) of Garg, Vazirani and Yannakakis [8] to directed networks, and has been previously applied by Klein et al. [13].

Lemma 6 ([8],[13]). Let G, c, x, and the k commodities be as above. Let r be any positive real and let q be any vertex of G. Then there exists a real number ρ, $0 < \rho \leq \ln(k+1)/r$, such that

$$c_E(B, V-B) \leq r \cdot (vol_E(B) + vol_E(V)/k),$$

where B denotes $B_G(q,\rho)$ (i.e., the set of vertices v such that G has a $q \to v$ path of length at most ρ). Moreover, there is an efficient algorithm to find ρ and $B_G(q,\rho)$.

3.3 An Algorithm and Proof For Theorem 2

Before describing the algorithm, we restate Theorem 2, for convenience.

Theorem 2. There is a polynomial-time algorithm that takes a k-commodity network G satisfying $c(e) \geq 1$ for all e and and finds a multicut M satisfying

$$c(M) \leq 39\ln(k+1)F(G)^2.$$

Proof of Theorem 2. Here is the algorithm:

Let $E' = E$, let $M = \emptyset$, and for each $i \in \{1,\ldots,k\}$, let $B_i = \emptyset$.

While there is a commodity $i \in \{1,\ldots,k\}$ such that $G' = (V, E')$ has an $s_i \to t_i$ path, **do**

Choose such an i.

Let $G_i = (V_i, E_i)$ be the subgraph of G' obtained by keeping exactly those vertices and arcs that belong to some $s_i \to t_i$ path in G'.

Apply Lemma 6 (the GVY procedure) to G_i with start vertex $q = s_i$ and $r = 3\ln(k+1)$, and let B_i be the vertex set given by the lemma, i.e., $B_i = B_{G_i}(s_i, \rho)$, where $\rho \leq \frac{\ln(k+1)}{r} = \frac{1}{3}$.

Add to M all the arcs of E_i in the cut $(B_i, V_i - B_i)$ in G_i.
Replace E' by $E' - M = E - M$.

End While

Output the multicut M and stop.

For the analysis, it is convenient to have $x(e) \leq 1/6$ for all arcs e. As in the proof of Theorem 1, we replace each arc e with $x(e) > 1/6$ by a path of $\lceil 6x(e) \rceil$ new arcs of length at most $1/6$ each, whose lengths add to $x(e)$, all of whose capacities are $c(e)$.

For each $i \in \{1, \ldots, k\}$ such that B_i is nonempty and for each vertex v in B_i, we assign a path of G_i, denoted $\sigma(i, v)$, and called the *suffix of v with respect to commodity i*. To define $\sigma(i, v)$, take any $s_i \to t_i$ path P of G_i that contains v, and let $\sigma(i, v)$ be the suffix of P of length at least $1/6$ and with the fewest vertices. Note that P exists (by our choice of G_i and the fact that $v \in B_i$) and has length at least 1 (by our choice of x). Clearly, $\sigma(i, v)$ has length less than $1/3$ (since $x(e) \leq 1/6$, $\forall e \in E$). Note that every vertex w in $\sigma(i, v)$ has $d_x(w, t_i) < 1/3$, and every vertex u in B_i has $d_x(s_i, u) \leq 1/3$; hence, $\sigma(i, v)$ is disjoint from B_i.

We need a claim.

Claim. Every vertex w of G is in at most $6F(G)$ sets B_i, $i \in \{1, \ldots, k\}$.

Proof of Claim. Focus on any vertex w and suppose that there are two commodities i and j such that w is in B_i and B_j. Assume without loss of generality that the algorithm processed i before j.

Suppose that $\sigma(i, w)$ and $\sigma(j, w)$ have a vertex y in common. Then G_j contains a $w \to y$ path called, say, P_j. Focus on G' at the start of the iteration for commodity i and call this digraph G^*. Clearly, G^* has an $s_i \to w$ path that is contained in B_i (since $w \in B_i$), G^* contains the $w \to y$ path P_j (since i is processed before j), and G^* has a $y \to t_i$ path that is a subpath of $\sigma(i, w)$. By concatenating these three paths, we see that G^* has an $s_i \to t_i$ walk W (allowing repeated vertices) that contains some arcs of P_j. Moreover, every arc of W that is in the cut $(B_i, V_i - B_i)$ in G^* is an arc of the middle path P_j, because the first of the three paths forming W has all its vertices inside B_i and the last of the three paths forming W has all its vertices outside B_i. Shortcut the $s_i \to t_i$ walk W to get an $s_i \to t_i$ path P in G^*. Then every vertex of P and arc of P is present in G_i. Moreover, in G_i, every arc of P in the cut $(B_i, V_i - B_i)$ is an arc of P_j, and there is at least one such arc. Hence, at least one arc of P_j is removed from E' by the iteration for commodity i. This is a contradiction, since P_j is supposed to be a path of G_j. Hence, $\sigma(i, w)$ and $\sigma(j, w)$ must be vertex-disjoint.

This proves the claim, since the suffixes $\sigma(i, w)$, where i is such that $w \in B_i$, are pairwise disjoint, and the number of suffixes is at most $\sum_e x(e)/(1/6) \leq 6 \sum_e c(e)x(e) \leq 6F(G)$, since each suffix has length at least $1/6$ and $c(e) \geq 1$ for all $e \in E$. ■

Clearly, the theorem holds if $k = 0$ or if $F(G) = 0$, since $C(G) = 0$ in these cases (the algorithm returns $M = \emptyset$). If $F(G) \neq 0$, then note that $F(G) \geq 1$, by the assumption on the capacities. The rest of the proof follows from Lemma 6 and the claim, since

$$
\begin{aligned}
c(M) &= \sum_{i=1}^{k} c_{E_i}(B_i, V_i - B_i) \\
&\leq \sum_{i=1}^{k} 3\ln(k+1)(\mathit{vol}_{E_i}(V_i)/k + \mathit{vol}_{E_i}(B_i)) \\
&\leq 3\ln(k+1)[F(G) + \sum_{i=1}^{k} \mathit{vol}_{E_i}(B_i)] \\
&\leq 3\ln(k+1)[F(G) + 12F(G)^2] \\
&\leq 39\ln(k+1)F(G)^2,
\end{aligned}
$$

where the inequality $\sum_{i=1}^{k} \mathit{vol}_{E_i}(B_i) \leq 12F(G)^2$ holds because

$$\sum_{i=1}^{k} \mathit{vol}_{E_i}(B_i) = \sum_{i=1}^{k} \sum \{c(u,v)x(u,v) \ : \ (u,v) \in E_i \text{ and } (u \in B_i \text{ or } v \in B_i)\} \ \leq$$
$$\sum_{(u,v)\in E} c(u,v)x(u,v)(\kappa(u)+\kappa(v)) \ \leq \ \sum_{(u,v)\in E} c(u,v)x(u,v)(12F(G)) \ \leq \ 12F(G)^2,$$

where $\kappa(v) \leq 6F(G)$ denotes the number of commodities $i \in \{1,\ldots,k\}$ such that vertex v is in B_i. ■

3.4 The Proof of Theorem 3

We restate Theorem 3, for convenience.

Theorem 3. There is a polynomial-time algorithm that takes an n-node, k-commodity network G and finds a multicut M satisfying

$$c(M) \leq (45\sqrt{n\ln(k+1)})F(G).$$

Proof of Theorem 3. The algorithm for Theorem 3 consists of two stages. Let $\alpha > 0$ be a parameter (later, we will fix $\alpha = 1/\sqrt{n\ln(k+1)}$).

In the first stage, we take M_1 to be the set of all arcs $e \in E$ such that $x(e) \geq \alpha$, and we take $E' = E - M_1$. M_1 is the subset of the multicut found by the first stage, and E' is the arc set of the current digraph after the first stage. (Informally, we "cut" all the arcs in M_1 by "rounding up" the LP solution x, and these arcs are ignored by the second stage.)

The second stage applies the algorithm of Theorem 2 to $G' = (V, E')$. Let M_2 be the multicut found by the second stage. The final multicut obtained by the algorithm is $M = M_1 \cup M_2$.

Consider the capacity $c(M)$ of M. First, $c(M_1) = \sum_{e\in M_1} c(e) \leq \sum_{e\in M_1} c(e)x(e)/\alpha \leq \sum_{e\in E} c(e)x(e)/\alpha$ $= F(G)/\alpha$, where the first inequality holds since the arcs e in M_1 have been "rounded up" from $x(e) \geq \alpha$ to 1.

We estimate $c(M_2)$ by modifying the analysis in the proof of Theorem 2 to exploit the fact that $x(e) < \alpha$ for all arcs e in the input. Choose any $i \in \{1,\ldots,k\}$ such that B_i is nonempty, let v be any vertex in B_i, and focus on the suffix $\sigma(i,v)$. Since $\sigma(i,v)$ has length at least $1/6$ and each arc e (in Stage 2) has $x(e) < \alpha$, there must be at least $(1/6)/\alpha$ vertices in $\sigma(i,v)$. By the claim in the proof of Theorem 2, any two distinct suffixes $\sigma(i,v)$ and $\sigma(j,v)$, $i \neq j$, are vertex-disjoint. Consequently, for any vertex v, the number of distinct suffixes $\sigma(i,v)$, $i \in \{1,\ldots,k\}$, is at most $n/(1/(6\alpha)) = 6\alpha n$. In other words, each vertex is in at most $6\alpha n$ distinct sets B_i, $i \in \{1,\ldots,k\}$. An argument similar to that of the proof of Theorem 2 implies $c(M_2)$ is at most $3\ln(k+1)[1+12\alpha n]F(G)$.

Then $c(M) = c(M_1) + c(M_2) \leq \dfrac{F(G)}{\alpha} + 3\ln(k+1)[1+12\alpha n]F(G)$. We balance the contribution of the two terms by choosing $\alpha = \dfrac{1}{\sqrt{n\ln(k+1)}}$ to get $c(M) \leq 3F(G)[\sqrt{n\ln(k+1)} + 14\sqrt{n\ln(k+1)}] =$ $(45\sqrt{n\ln(k+1)})F(G)$. ■

Remarks. (1) Theorem 3 imposes no restrictions on the arc capacities. (2) Theorem 3 implies that the integrality ratio of the linear program is at most $45\sqrt{n\ln(k+1)}$, and hence any network with integrality ratio at least $k/2$ must have $n \geq k^2/(90^2\ln(k+1))$.

3.5 Bounded-Degree Planar Digraphs

In this section we prove Theorem 4, which is restated here for convenience.

Theorem 4. For every Δ, there are a c and a polynomial-time algorithm that takes any n-node, k-commodity network G with $k \geq 2$, having uniform capacities, in which the total degree of every vertex is at most Δ and whose underlying undirected graph is planar, and finds a multicut M in G satisfying $c(M) \leq (c\sqrt{\lg k})n^{1/4}F(G)$.

We use the following planar separator lemma:

Lemma 7 (implicit in Lipton and Tarjan [17]). For every integer $\Delta > 0$ there exists a constant $\alpha = \alpha(\Delta) \geq 1$ such that for every (undirected) planar multigraph $G = (V, E)$ with maximum degree at most Δ, there are disjoint subsets $L, R \subseteq V$ of size $\lfloor |V|/2 \rfloor$ each, and a set Z of edges of size at most $\alpha\sqrt{|V|}$, such that every edge not in Z either has both endpoints in L or both in R. Furthermore, there is a polynomial-time algorithm that finds such a set of edges.

Proof of Theorem 4. By Theorem 2, there is a universal constant β such that the multicut size in a uniform-capacity network G is at most $(\beta \lg k)F(G)^2$, if $k \geq 2$.

Fix Δ. We prove that for every $n \geq 1$, for every $k \geq 2$, in every uniform-capacity, k-commodity, n-node network G on a planar digraph the total degree of each vertex of which is at most Δ, there is a multicut of size at most $(c\sqrt{\lg k})n^{1/4}F(G)$, where $c = \max\{\frac{\alpha}{1-2^{-1/4}}, \beta\}$. The proof can easily be converted into a polynomial-time algorithm.

Our proof proceeds by induction on n. The basis of the induction ($n = 1$) is trivial. Consider an n-node instance G, $n \geq 2$. If $F(G) \leq n^{1/4}/\sqrt{\lg k}$, then by Theorem 2, the minimum multicut is of size at most $(\beta \lg k)F(G)^2 \leq \beta\sqrt{\lg k} \cdot F(G)n^{1/4} \leq c\sqrt{\lg k} \cdot F(G)n^{1/4}$. So, we may assume that $F(G) > n^{1/4}/\sqrt{\lg k}$.

By Lemma 7, we can find a set of at most $\alpha\sqrt{n}$ arcs whose removal partitions G into two subgraphs of order $\lfloor n/2 \rfloor$ each (with perhaps one isolated vertex left over). Clearly, every commodity with terminals in two different components is cut by removing the at-most-$\alpha\sqrt{n}$ arcs. Let f_1, f_2 be the maximum flows for the remaining commodities in the two components. By the inductive assertion, for $i = 1, 2$ one can find a multicut of size at most $(c\sqrt{\lg k})f_i\lfloor n/2 \rfloor^{1/4}$ in the ith component. (This holds even if either component has 0 or 1 commodity, though this case isn't covered by the inductive assertion.) The union of these multicuts and the separator is a multicut for the entire instance. Its size is at most

$$\begin{aligned}
\alpha\sqrt{n} + c\sqrt{\lg k} \cdot \sum_{i=1}^{2} f_i(n/2)^{1/4} &\leq \alpha\sqrt{n} + c\sqrt{\lg k} \cdot (f_1 + f_2)(n/2)^{1/4} \\
&\leq (\alpha + c/2^{1/4})\sqrt{\lg k} \cdot F(G)n^{1/4} \\
&\leq c\sqrt{\lg k} \cdot F(G)n^{1/4},
\end{aligned}$$

as $f_1 + f_2 \leq F(G)$, $F(G) > n^{1/4}/\sqrt{\lg k}$, and $c \geq \alpha + \frac{c}{2^{1/4}}$. ∎

4 Some Simple Constructions Must Be Large

In this section we prove that certain arc-capacitated or node-capacitated networks with k commodities and integrality ratio at least $k/2$ must have exponentially many vertices; any n-vertex network with integrality ratio at least n^ϵ (for a fixed $\epsilon > 0$) must exploit the "directedness" of the network more than the Saks et al. example does.

Theorem 8. Let $H' = (V', E')$ be an undirected graph in which each edge has some capacity $c(e) > 0$. Replace each edge e by a pair of antiparallel arcs each of capacity $c(e)$ and call the resulting digraph $H = (V', E)$. Add $2k$ distinct new vertices $s_1, t_1, ..., s_k, t_k$, getting vertex set $V = V' \cup \{s_1, t_1, ..., s_k, t_k\}$. Choose subsets $S_i \subseteq V', T_i \subseteq V'$, $S_i \cap T_i = \emptyset$ for all $i = 1, 2, ..., k$ and add arcs (s_i, v) for all $v \in S_i$ and (u, t_i) for all $u \in T_i$, all of infinite (or very large) capacity. Where G is the resulting network,

$$C(G) \leq (4c \lg n) F(G),$$

where c is a universal constant and $n = |V'|$.

First, a simple corollary.

Corollary 9. Using the notation of Theorem 8, if G has integrality ratio at least $k/2$, then $n \geq 2^{k/(8c)}$.

Proof of Theorem 8. Starting with digraph H, define a network on V' by constructing one commodity for each pair (u, v) with $u \in S_i$ and $v \in T_i$ for some i, having source u and sink v; call this network H also. The key point is that $F(H) = F(G)$ and $C(H) = C(G)$.

We now construct an undirected version of H and apply the result of [8] on integrality ratios of undirected networks. Build an undirected network called H' by starting from undirected graph H' and defining a commodity for every unordered pair $\{u, v\}$ such that $u \in S_i, v \in T_i$ for some i. We have $C(H) \leq 2C'(H')$ and $F(H) \geq F'(H')$. Apply [8] to infer that there is a universal c such that

$$C'(H') \leq \left(c \lg \binom{n}{2} \right) F'(H') \leq (2c \lg n) F'(H')$$

and then conclude that

$$C(G) = C(H) \leq 2C'(H') \leq (4c \lg n) F'(H') \leq (4c \lg n) F(H) = (4c \lg n) F(G). \blacksquare$$

Now we state a node-capacitated version. A *node multicut* in a node-capacitated digraph G is a subset of vertices containing at least one vertex on every $s_i \to t_i$ path, for all i. However, to discourage deletion of terminals, we insist that all terminals have infinite capacity. Similarly, a *node multicut* in a node-capacitated undirected graph G' is a subset of vertices containing at least one vertex on every $s_i - t_i$ path, for all i. Again we insist that all terminals have infinite capacity. Let $NC(G), NC'(G')$ denote the minimum capacity of a node multicut in digraph G or undirected graph G', respectively.

There is an obvious LP relaxation with nonnegative variables $x(v)$ for all $v \in V$, constrained so that for all i, all $s_i \to t_i$ paths P satisfy $\sum_{v \in P} x(v) \geq 1$, whose objective is the minimization of $\sum c(v)x(v)$; for undirected graphs, we have the same problem, except involving undirected $s_i - t_i$ paths. The corresponding duals are flow problems: Find a nonnegative value for each $s_i \to t_i$ (or $s_i - t_i$) path, for all i, such that the sum of the values on all paths containing vertex v is at most $c(v)$, and maximize the sum of all variables. Let $NF(G), NF'(G')$ be the maximum flow value in digraph G or undirected graph G', respectively.

Garg, Vazirani, and Yannakakis [9] prove a node analogue to their arc result: There is a c such that $NC'(G') \leq (c \lg k) NF'(G')$ for all G' with $k \geq 2$ commodities.

Theorem 10. Let $H' = (V', E')$ be an undirected graph in which each node v has some capacity $c(v) > 0$ (and edges are uncapacitated). Replace each edge e by a pair of antiparallel arcs and call the resulting digraph $H = (V', E)$. Add $2k$ distinct new vertices $s_1, t_1, ..., s_k, t_k$ having infinite capacities, getting vertex set $V = V' \cup \{s_1, t_1, ..., s_k, t_k\}$. Choose subsets $S_i \subseteq V', T_i \subseteq V'$, $S_i \cap T_i = \emptyset$ for all $i = 1, 2, ..., k$ and add arcs (s_i, v) for all $v \in S_i$ and (u, t_i) for all $u \in T_i$. Where G is the result,

$$NC(G) \leq (4c \lg n) NF(G),$$

where c is a universal constant and $n = |V'|$.

The Saks et al. construction is of this form.

Corollary 11. Using the notation of Theorem 10, if G has integrality ratio at least $k/2$, then $n \geq 2^{k/(8c)}$.

The proof of Theorem 10 is so similar to that of Theorem 8, we omit it, except to note that we make the terminals of H and H' *new* vertices (of infinite capacity) outside of V', since effectively we cannot delete any terminal.

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