Topological Methods - Exercise 3

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1. Let Δ be a k-dimensional simplex and T a triangulation of Δ. Recall that if $x \in \Delta$, then $supp(x)$ is the vertex set of the lowest-dimensional face of Δ that contains x. Let the vertex set of Δ be $\{v_1, \ldots, v_{k+1}\}$, and label v_i by i. EXtend this to any labelling of $V(T)$ such that every $x \in V(T)$, is labelled by an element from the label set of $supp(x)$.

   Sperner's Theorem says that in every such situation there is a *full* simplex in T, namely a simplex whose vertices exhibit all labels $\{1, \ldots, k+1\}$.

   Use the proof technique shown in class for Tucker's Lemma to prove Sperner's Theorem.

2. Let P^{n-1} denote the (abstract) simplicial complex with vertex set $V(P^{n-1}) = \{\pm 1, \pm 2, \ldots, \pm n\}$, and with a subset $F \subset V(P^{n-1})$ forming a simplex whenever there is no $i \in [n]$ such that both $i \in F$ and $-i \in F$. Prove that the following is an equivalent reformulation of Tucker's lemma

   Lemma 1 *Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Then there is no map $\lambda : V(T) \to V(P^{n-1})$ that is a simplicial map of T into P^{n-1} and is antipodal on the boundary.*