Topological Methods in Combinatorics - Exercise 1

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1. Let $v_0, v_1 \ldots, v_k \subset \mathbb{R}^d$. We say that $\{v_i\}$ are affinely dependent if there are real numbers $\alpha_1, \ldots, \alpha_k$ such that $\sum_{i=1}^k \alpha_i v_i = 0$ and $\sum_{i=1}^k \alpha_i = 0$. Otherwise $\{v_i\}$ are called affinely independent. Show that the following two conditions are equivalent to affine independence

   - The k vectors $v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0$ are linearly independent.
   - The vectors $(1, v_0), (1, v_1), \ldots, (1, v_k) \in \mathbb{R}^{d+1}$ are linearly independent.

2. Determine the faces of the d-dimensional crosspolytope.

3. Verify the following homeomorphisms:

   - $\mathbb{R} \cong (0,1) \cong S^1 \backslash \{(0,1)\}$
   - $S^1 \cong \partial([0,1]^2)$
   - Let A and B be two convex bodies in $\mathbb{R}^d$. Show that $A \cong B$, $\partial(A) \cong \partial(B)$, and $int(A) \cong int(B)$.

4. Let X and Y be topological spaces.

   - Show that a mapping $f : X \to Y$ is continuous iff $f^{-1}(F)$ is closed for each closed set $F \subset Y$.
   - Let X be covered by finitely many closed sets $A_1, \ldots, A_n$. Show that if a mapping $F : X \to Y$ restricted to each A_i is continuous then f is continuous.

5. Show that Homotopy equivalence as defined in class is indeed an equivalence relation among topological spaces.

6. Let X be a topological space and $Y \subset X$ a subspace. A deformation retraction of X onto Y is a family $\{f_t\}_{t \in [0,1]}$ of continuous maps $f_t : X \to X$ such that

   - $f_0 = I_X$,

- $f_t(y) = y$ for all $y \in Y$ and $t \in [0,1]$, and
- $f_1(X) = Y$,

The mappings should depend continuously on t. If a deformation retraction as above exists, Y is called a deformation retract of X. Prove that if there exists a space Z such that both X and Y are a deformation retracts of Z then X and Y are homotopy equivalent.

7. Prove that the set of all faces of a simplex is a simplicial complex.

*8. The chessboard complex $Ch_{m,n}$ has the mn squares of the $m \times n$ chessboard as vertices, and simplices are all subsets of squares such that no two squares lie in the same row or column. Describe the "geometric shape" of of $\|Ch_{3,4}\|$.