Q-warping: Direct Computation of Quadratic Reference Surfaces

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Abstract

We consider the problem of wrapping around an object, of which two views are available, a reference surface and recovering the resulting parametric flow using direct computations (via spatio-temporal derivatives). The well known examples are affine flow models and 8-parameter flow models — both describing a flow field of a planar reference surface. We extend those classic flow models to deal with a Quadric reference surface and work out the explicit parametric form of the flow field. As a result we derive a simple warping algorithm that maps between two views and leaves a residual flow proportional to the 3D deviation of the surface from a virtual quadric surface. The applications include image morphing, model building, image stabilization, and disparate view correspondence.

1 Introduction

The image flow field induced by the camera (or scene) motion is a product of the 3D structure of the scene, the 3D camera motion parameters and its intrinsic parameters. While efficient factorization of the flow into structure and motion is an active area of research, in many cases an implicit parametric form of the optic flow is sufficient, say for motion segmentation, image stabilization, coarse correspondence for model building, and for establishing reference surfaces for 3D scene representation. To that end, a hierarchy of parametric flow models have been developed in the past (whose most elegant description can be found in [3]) starting from pure global translation, image plane rotation, 2D affine, and 2D homography (8-parameter flow, also known as quadratic flow). These models have been used extensively and have been estimated directly from image spatio-temporal derivatives (known as *direct estimation*) using coarse-to-fine estimation via Laplacian (or Gaussian, or wavelets) pyramids. These methods search for the best (parametric) flow model out of the family of constrained flows (described above) that minimizes the square of change of image intensities (SSD) over the whole image — thus gaining robustness due to very highly over-constrained linear systems (each pixel contributes a linear constraint).

However, these models are applicable to scenes that are approximately planar or have small variations in depth, relative to the distance from the camera. The residual flow, followed by the nominal planar warp, is proportional to the depth variation from the scene to the virtual planar surface. In other words, the virtual surface plays a role of a *reference surface* and the residual flow represents the scene structure relative to the reference surface. In many of the applications mentioned above one would like the reference surface to approximate the structure of the scene. For example, a planar scene with a small number of objects protruding from the plane (such as a moving vehicle) is ideal for an affine or homography flow that will stabilize the plane and thereby enhance the position of the protruding objects. Small residual flow is also convenient for establishing correspondence (alignment) between disparate views. Since the nominal flow (corresponding to the parametric flow model) is highly over-constrained large image distances can be tolerated, and if the residual flow is small then a second round of optic flow (now unconstrained) can handle the remaining displacements.

These examples naturally suggest considering higher-order parametric flows in order to account for non-planar virtual reference surfaces. For example, by placing a virtual quadric surface (allowing for all degenerate forms including a plane) on the object would give rise to a smaller residual flow and in more general circumstances. Consider for example Fig. 2 displaying two widely separated views of a face. Notice the effect of a planar warp field, compared to the effect of a Quadric-based flow field. The image warped by the nominal flow is much less distorted and the residual flow is much smaller.

The idea of extending the planar models to quadric models was originally suggested in [13] but in the context of discrete motion. The quadric reference surface was recovered using explicit point matches, including the computation of the epipolar geometry, whereas here we wish to establish a "quadric warp field" using infinitesimal motion models and direct estimation. In [13] special attention was payed on how to overcome the multiple solution problem since a general ray from the camera meets a quadric twice, thus for every pixel in the first image there would be two candidate projections in the second image. Also special attention was payed to the type of image measurements that are sufficient for a solution (point matches only, points and an outline conic, see also [6]).

In this paper we introduce the derivation of a quadric-based nominal flow field, we call *Q-warping*, using the infinitesimal motion model and direct estimation. The multiple solution problem addressed by [13] via "opacity" constraint is approached differently here. Instead of using an opacity assumption, which is problematic to enforce in a parametric flow representation, we enforce the family of quadrics to include the center of projection of the first camera. We show that the assumption does not reduce the generality of the approach due to existence of hyperboloids of two-sheet (one sheet includes the camera center and the other wraps around the object), and that planar surfaces are included in this model in a general manner, i.e., the plane may be generally located in space. Therefore, our model extends the hierarchy of direct estimation parametric models of [3] without sacrificing "backward compatibility".

2 Background: Small Motion and Parametric Flow

The parametric flow models are based on combining three elements: (i) infinitesimal motion model, (ii) planar surface assumption being substituted into the motion model, and (iii) the parametric flow is integrated with the "constant brightness constraint". We will describe these elements in detail below.

2.1 Small Motion Model

We describe below a compact form of the Longuett-Higgins and Prazdny motion model [10]. Let $p = [x, y]^t = [X/Z, Y/Z]^t$ where $P = [X, Y, Z]^t$ is a world point in the coordinate system of the first (calibrated) camera and p is it's corresponding image point. Let P' be the coordinates of the same world point in the second camera coordinate frame. Since the camera motion is rigid, we have $P' = RP + t$ where R and t are the rotation and translation between the coordinate frames.

The rotation matrix R can be written as

$$R = I * cos\phi + (1 - cos\phi)ww^t + sin\phi * [\omega]_\times$$

where ω is a unit vector representing the screw axis, $[\omega]_x$ is the skew-symmetric matrix of vector products, i.e., $[\omega]_x v = \omega \times v$ for all vectors v, and ϕ is the angle of rotation around the screw axis. When ϕ is small, $cos\phi \rightarrow 1$ and $sin\phi \rightarrow \phi$, and in turn $R = I + [\omega]_\times$ where the magnitude of ω is the angle of rotation. Given the instantaneous rotation, the instantaneous motion of P is:

$$\begin{aligned} \dot{P} &= \frac{dP}{dt} \approx P' - P = RP + t - P = \\ &= (I + [\omega]_\times)P + t - P = [\omega]_\times P + t \end{aligned} \quad (1)$$

Let $[u, v]^t$denote the image velocity from p to p'. We use $\dot{X}$ and $\dot{Z}$, as defined by (1) to get the following:

$$\begin{aligned} u &= \frac{dx}{dt} = \frac{d}{dt}(\frac{X}{Z}) = \\ &= \frac{\dot{X}Z - X\dot{Z}}{Z^2} = \frac{1}{Z}(\dot{X} - x\dot{Z}) = \\ &= \frac{1}{Z}[1,\ 0, -x]^t([\omega]_\times P + t) \end{aligned}$$

and $v = \frac{dy}{dt}$ is similarly derived. To summarize we have,

$$\begin{aligned} u &= \frac{1}{Z}s_1^t t + s_1^t[\omega]_\times p \\ v &= \frac{1}{Z}s_2^t t + s_2^t[\omega]_\times p \end{aligned} \quad (2)$$

where $s_1 = [1, 0, -x]^t$ and $s_2 = [0, 1, -y]^t$.

2.2 Direct Estimation Equation

Assume the brightness constancy assumption,

$$I'(x, y) = I(x - u, y - v),$$

where $I(x, y), I'(x, y)$ are the observed grey-scale images at two successive time frames. Since the displacement u, v are assumed to be small (infinitesimal motion assumption), then the equation above can be simplified through the truncated (first-order) Taylor series expansion of $I(x, y)$ to what is known as the "constant brightness equation" [8]:

$$uI_x + vI_y + I_t = 0$$

where I_x, I_y are the x, y spatial derivatives, respectively, and $I_t = I'(x, y) - I(x, y)$ is the temporal image derivative. By substituting u, v with equations 2 we obtain a linear constraint on the flow u, v as a function of the spatio-temporal derivatives, the camera motion parameters ω, t and the depth variable Z:

$$\frac{1}{Z}(I_\times s_1 + I_y s_2)^t t + (I_x s_1 + I_y s_2)^t[\omega]_\times p + I_t = 0,$$

which after simplification becomes [9]:

$$\frac{1}{Z}s^t t + s^t[\omega]_\times p + I_t = 0 \quad (3)$$

where

$$s = \begin{bmatrix} I_x \\ I_y \\ -xI_x - yI_y \end{bmatrix}.$$

2.3 Parametric Model: Planar Case

One can eliminate the parameter Z from equation 3 by assuming that the scene is planar [1], i.e., there exist scalars A,B,C such that $AX + BY + CZ = 1$ for all points $[X, Y, Z]^t$. By definition of p we have $X = xZ, Y = yZ$ and so we can rewrite it as $\frac{1}{Z} = Ax + By + C$. Substituting this in (2) yields:

$$\begin{aligned} u &= (Ax + By + C)s_1^t t + s_1^t[\omega]_\times p \\ v &= (Ax + By + C)s_2^t t + s_2^t[\omega]_\times p \end{aligned}$$

Written more explicitly, let $t = [\tau_1, \tau_2, \tau_3]^t$ and $\omega = [\alpha, \beta, \gamma]^t$ we get:

$$\begin{aligned} u &= (A\tau_1 - C\tau_3)x + (B\tau_1 - \gamma)y + C\tau_1 + \beta - \\ &\quad (B\tau_3 + \alpha)xy + (\beta - A\tau_3)x^2 \\ v &= (A\tau_2 + \gamma)x + (B\tau_2 - C\tau_3)y + C\tau_2 - \alpha + \\ &\quad (\beta - A\tau_3)xy - (B\tau_3 + \alpha)y^2 \end{aligned}$$

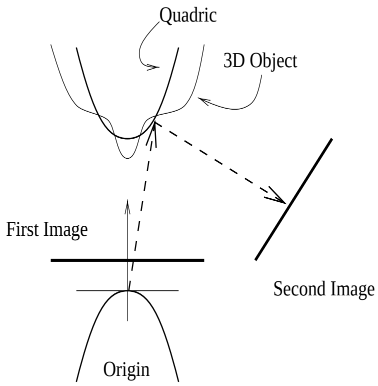


Figure 1. Q-warping fits a quadric around a general object, however the family of quadrics must contain the origin (the first camera center). A hyperboloid of two sheets meets this requirement by having one sheet coincide with the origin and the other sheet wrap around the object. All rays from the first camera intersect the quadric uniquely and the projection of the intersection point onto the second view is the result of the Q-warping flow. The residual flow is thus proportional to the deviation of the physical surface from the virtual quadric.

The terms above can be collected to give the 8-parameter flow model used for estimating the instantaneous motion of a plane:

$$\begin{aligned} u &= ax + by + c + gxy + hx^2 \\ v &= dx + ey + f + hxy + gy^2 \end{aligned} \quad (4)$$

The direct estimation readily follows by substituting the above in $uI_x + vI_y + I_t = 0$ we obtain a linear constraint on the parameters $a, b, ..., h$. Every pixel with a non-vanishing gradient contributes one linear constraint for 8 unknowns, thus, making a highly over-constraint least-squares system for solving the warping function 4.

3 The Quadric Flow: Q-warping

Consider the family of quadric surfaces that contains the origin $[0, 0, 0]$, i.e., the center of projection of the first camera:

$$\begin{aligned} AX + BY + Z + DXY + EXZ + FYZ + \\ +GX^2 + HY^2 + KZ^2 = 0 \end{aligned} \quad (5)$$

Note that we have normalized the coefficients assuming that the coefficient of Z is non-vanishing, one could choose other forms of normalization.

The reason we include the origin is to have a single intersection between the optical rays emanating from the first camera and the quadric surface. A single intersection is a necessary condition for obtaining a warping function. It is important to note that the inclusion of the origin does not limit the generality of the quadric because quadrics can break apart into two pieces, known as the hyperboloid of two sheets. Thus, one sheet will include the origin and the other sheet will wrap around the object. The location and shape of the sheet (parabolic, elliptic, and degenerate forms like cylinders and cones) will be determined by the image measurements of spatio-temporal derivatives (see Fig. 1). What is also important to show is that the inclusion of the origin is not a constraint that is carried to the degenerate form of a planar surface. In other words, a planar warping function, corresponding to any general position of a plane, should be a particular case of the Q-warping function — otherwise we will not be able to include Q-warping in the hierarchy of parametric models. We will show later that in case of planar objects, the quadric breaks down into two planes, one coincides with the physical plane in the scene and the other is the plane $Z = 0$. Taken together, there is no loss of generality by having the origin live inside the family of quadrics.

Using $X = xZ, Y = yZ$ and dividing by Z^2 and rearranging terms, we get:

$$\frac{1}{Z} = \frac{Dxy + Ex + Fy + Gx^2 + Hy^2 + K}{Ax + By + 1}$$

By substituting Z in equations 2 we obtain a parametric flow model (with 17 distinct parameters), which is our Q-warping function:

$$u = \frac{\alpha(x, y, a, ..., p)}{Ax + By + 1} \quad (6)$$

$$v = \frac{\beta(x, y, a, ..., p)}{Ax + By + 1} \quad (7)$$

where,
$\alpha(\cdot) = ax + by + c + dxy + ex^2 + fy^2 + gyx^2 + hxy^2 + px^3$
$\beta(\cdot) = jx + ky + l + mxy + nx^2 + oy^2 + pyx^2 + gxy^2 + hy^3$

And:

$a = \beta A - K\tau_3 + E\tau_1$ $\quad b = \beta B - \gamma + F\tau_1$
$c = K\tau_1 + \beta$ $\quad d = -\alpha - D\tau_1 - \gamma A - F\tau_3$
$e = \beta + G\tau_1 - E\tau_3$ $\quad f = H\tau_1 - \gamma B$
$g = \beta B - D\tau_3 - A\alpha$ $\quad h = -H\tau_3 - B\alpha$
$j = \gamma - A\alpha + E\tau_2$ $\quad k = F\tau_2 - K\tau_3 - B\alpha$
$l = K\tau_2 - \alpha$ $\quad m = B\gamma + \beta + D\tau_2 - E\tau_3$
$n = G\tau_2 + A\gamma$ $\quad o = H\tau_2 - \alpha - F\tau_3$
$p = \beta A - G\tau_3$

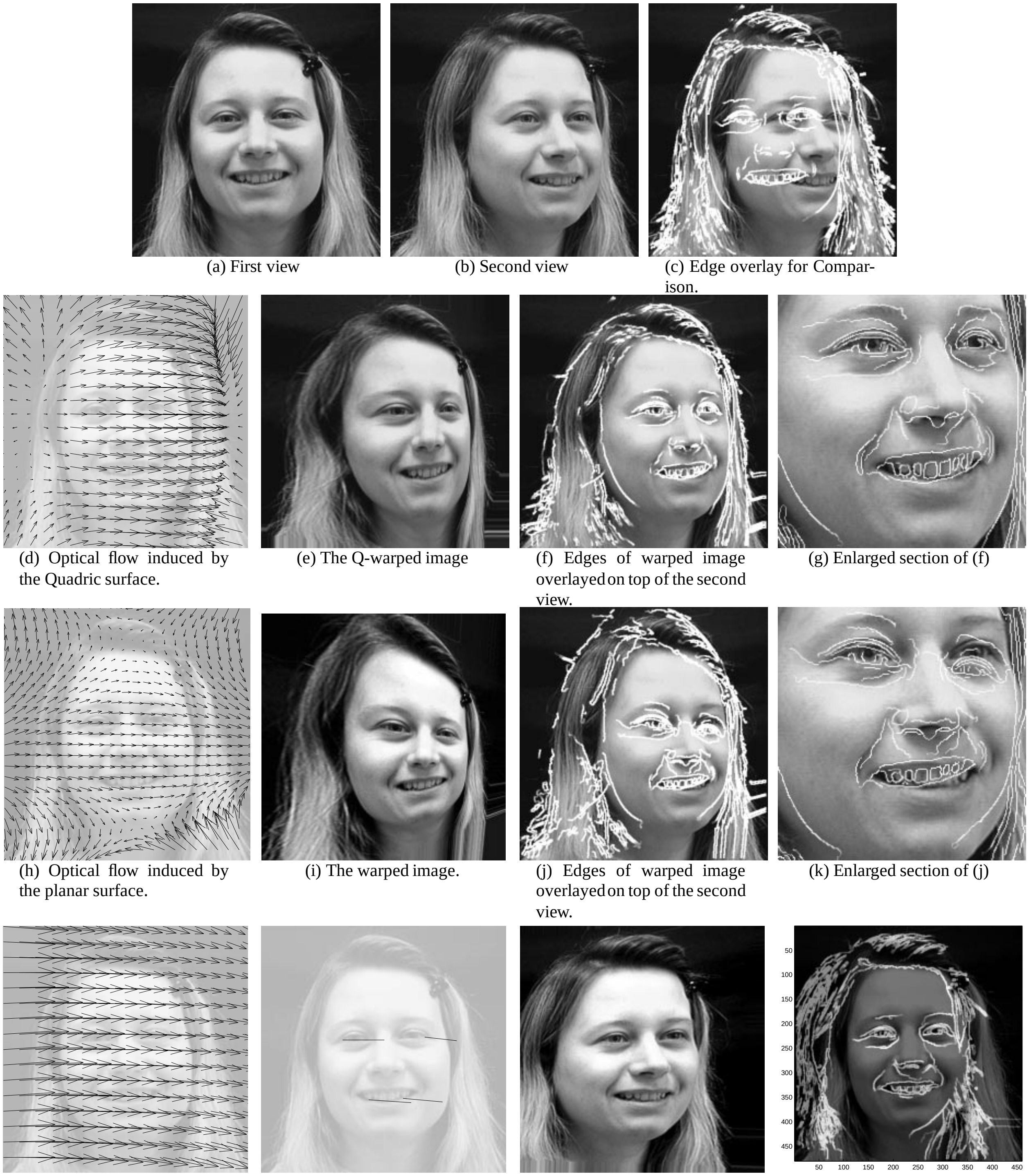
Before we continue to the direct estimation equation, consider the case of a planar object. We would like to show the following:

Proposition 1 *The Q-warping flow model includes as a particular case the planar parametric model of equations 4.*

Proof: Consider the quadric: $EXZ + FYZ + KZ^2 + Z = 0$. By dividing by Z^2 followed by substitution in equations 2, we obtain:

$$\begin{aligned} u &= ax + by + c + dxy + ex^2 \\ v &= jx + ky + l + mxy + oy^2 \end{aligned}$$

where $d = o = -\alpha - F\tau_3$ and $e = m = \beta - E\tau_3$. $\square$



(a) First view (b) Second view (c) Edge overlay for Comparison.

(d) Optical flow induced by the Quadric surface. (e) The Q-warped image (f) Edges of warped image overlayed on top of the second view. (g) Enlarged section of (f)

(h) Optical flow induced by the planar surface. (i) The warped image. (j) Edges of warped image overlayed on top of the second view. (k) Enlarged section of (j)

(l) Optical flow (m) matches on 1st image (n) Warped image (o) Edges of warped on 1st image

Figure 2. Application of Q-warping on general objects. *Row 1* displays the original two views and the edge overlay in order to appreciate the distance between matching features. *Row 2* displays the Q-warping results. Note that the features are aligned up-to a few pixels. The alignment is not expected to be accurate because the object is not a quadric, but the small residual flow suggests that the fitted quadric was wrapped closely around the object. *Row 3* compares the results with a direct estimation planar flow (eqn. 4). *Bottom Row* compares the results with a planar flow recovered from discrete 4 point matches from the center of the face.

In order to obtain a direct estimation using spatio-temporal derivatives, we multiply both sides of the Q-warping equations by $Ax + By + 1$ and obtain:

$$\alpha(x, y, a, ..., p)I_x + \beta(x, y, a, ..., p)I_y + (Ax + By + 1)I_t = 0. \quad (8)$$

This is a linear equation in $A, B, a, b, ..., p$ per pixel with non-vanishing gradients. The least-squares estimation requires some care which will be described next. Also note that the motion model has 17 parameters, yet the minimal number of parameters required for describing a moving quadric passing through the camera center is $14 = 6 + 8$, where 6 comes from parameters of rotation and translation and 8 comes from the number of parameters representing the quadric (passing through the origin). Therefore, the 17 parameters must satisfy algebraic constraints, i.e., not every set of 17 numbers is admissible. However, this is a topic which will not be covered in the scope of this paper.

4 Iterative Refinement

The parametric flow models are only first-order approximations, and therefore, the implementation framework must include a Newton iterative refinement, of the style suggested in [2, 3, 4, 11]. The iterative refinement process gradually brings the two original images closer to each other, such that in the ideal case of constant brightness and a quadric surface, $I_t \rightarrow 0$ at the limit. Considering the estimation equation 8, the diminishing I_t is a serious problem because the coefficients A, B become under-determined. In other words, as we get closer to the solution, our system of equations gets increasingly unstable numerically.

We adopt the line of approach described in [7, 15] which is to rewrite the direct estimation equation 8 as a function of the final output flow instead of the incremental flow, shown next. Let $u(x, y), v(x, y)$ be the final flow (describing the displacement field between the original two images $I(x, y), I'(x, y)$ as a function of a quadric model) described parametrically in eqns. 6 and 7.

Let $\tilde{u}(x, y), \tilde{v}(x, y)$ be the flow field established in the last iteration (the initial guess for the current iteration). The incremental flow is defined by $\Delta u = u - \tilde{u}$ and $\Delta v = v - \tilde{v}$ satisfies the constant brightness equation: $\Delta u I_x + \Delta v I_y + I_t = 0$. After substitution we obtain the new direct estimation equation for the parameters $A, B, a, ..., p$ below:

$$\alpha(\cdot)I_x + \beta(\cdot)I_y + (Ax + By + 1)(I_t - \tilde{u}I_x - \tilde{v}I_y) = 0. \quad (9)$$

Initially, $\tilde{u} = \tilde{v} = 0$. As the iterations proceed $\tilde{u}, \tilde{v}$ approach the desired flow u, v (eqns. 6,7). In each iteration, the system of equations for the parameters $a, ..., p, A, B$ is defined by minimizing the least squared error:

$$\begin{aligned} Err &= \sum_{x,y} = [\alpha(\cdot)I_x + \beta(\cdot)I_y \\ &+ (Ax + By + 1)(I_t - \tilde{u}I_x - \tilde{v}I_y)]^2 \end{aligned} \quad (10)$$

where the sum is over the entire image. The system of linear equations is obtained by setting the partial derivatives of (10) with respect to each of the parameters $a, ..., p, A, B$ to zero.

The above estimation algorithm is meaningful only when the frame-to-frame displacements are a fraction of a pixel so that the first-order order term of the Taylor series is dominant. The range of displacements can be extended to large displacements by implementing the procedure within a multiresolution (pyramid) structure [5, 12]. Further details on the iterative refinement, warping, and coarse-to-fine implementation can be found in [14].

5 Experiments

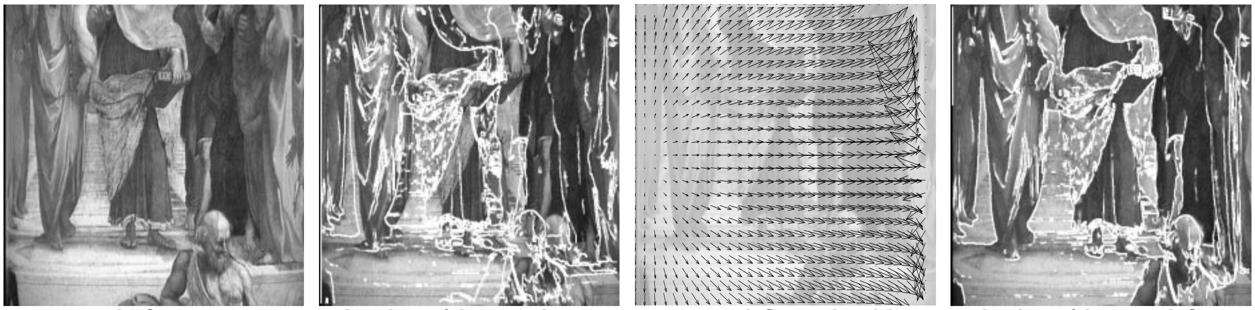
We have conducted a number of experiments both on quadrics and general objects. In our first example we have wrapped a poster on an approximately cylindrical surface. Here we expect the parametric flow recovered from the Q-warping function to match closely the true flow. Fig. 3 displays the results: the edges of the second view are overlayed on the first view in order to visualize the magnitude of displacements. The flow field recovered by our algorithm is displayed in Fig. 3c and the edges of the warped image overlayed on the second view shown in Fig. 3d match their true position up to sub-pixel accuracy. Note that this also demonstrates that the real surface need not contain the center of projection of the first camera because the Q-warping function can feet a hyperboloid of two sheets with one of the sheets wrapping around the physical surface.

The next example is on a general object. Fig. 2 shows two disparate views of a face. One can see from the edge overlay that the distance between the views is fairly significant. Note the difference between the Q-warping and the planar warping. We have applied to versions of a planar warping. First, in the bottom row 4 matching points were selected (coming from an approximately planar configuration in space) and the 2D projective transformation determined by 4 matching points) was recovered. Note that the edges on the center of the face are closely aligned at the expense of the boundary curves (as the boundary curves away from the plane determined by the 4 matching points). Second, we applied the infinitesimal planar motion model (eqn. 4) in a direct estimation framework (row 3). Note that the planar warping has chosen a plane fitting the center of the face (where most of the strong gradients are) rotated around the vertical axis — the result is a distorted warped image due to the large deviation of the object from a planar surface. The Q-warping on the other hand has aligned the warped image with the second view, up-to a few pixels distance.

Additional experiments can be found in [14]. Taken together, the Q-warping algorithm generates a parametric flow field that performs well on general objects as well as on quadric surfaces (in the latter case the flow field is exact).

6 Summary

We have extended the parametric flow hierarchy to include flows induced by a virtual quadric. We have shown that the extension can be made feasible in the sense that the warping function is unique and includes planar warping as a particular case, when the family of quadrics contain the center of projection of the first camera. We have shown that containing the origin does not limit the generality of the quadric fitting and



(a) The first view. (b) Edges of the second view overlayed on the first. (c) Optical flow induced by the quadric surface (d) Edges of the warped (first) view overlayed on the second.

Figure 3. A poster was wrapped around a cylindrical surface. The recovered flow matches the true flow to sub-pixel accuracy.

have proven that the planar case is included within the model. Experiments on real images of general objects illustrate the applicability of our method. Q-warping provides more flexibility in fitting flow fields to the scene than existing planar models, yet reduces to planar warping when the scene requires so.

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