Ambiguity in Reconstruction From Images of Six Points

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Abstract

Let S be a set of six points in space, let ψ be any hyperboloid of one sheet containing S, and let I be a sequence of images of S taken by an uncalibrated camera moving over ψ. Then reconstruction from I is subject to a three way ambiguity which is unbroken as long as the optical centre of the camera remains on ψ.

Let $\mathbf{p}$ be an image of S taken from a point on ψ. The images 'near' $\mathbf{p}$ define a tangent space which splits into a direct sum $W_p \oplus N_p \oplus F_p$, where W_p corresponds to images near $\mathbf{p}$ for which the ambiguity is maintained, N_p corresponds to images for which the ambiguity is broken and F_p corresponds to images which are physically impossible.

1 Introduction

The geometry underlying reconstruction from pairs of images has been studied in great detail over the past 20 years [3], [4], [8]. Much less is known about reconstruction from three or more images. General frameworks for reconstruction from multiple images are described in [5], [6], [7], [13], and more detailed results are given in [9], [11]. Experimental results are reported in [1], [6], [9], [11], [12].

It is shown in [9] that reconstruction from three images of six points is subject to a three way ambiguity. Our main result is the surprising fact that the three way ambiguity is preserved as long as the optical centre of the camera remains on a certain quadric surface ψ; the ambiguity is broken only if the optical centre leaves ψ.

We also investigate the problem of finding the best camera motion to break the ambiguity. To first order, any camera motion starting at a point $\mathbf{x}$ of ψ and not tangential to ψ breaks the ambiguity. At first sight it looks as if a camera motion normal to ψ at $\mathbf{x}$ is the best choice. Unfortunately ψ is defined only up to a collineation of $\mathbb{P}^3$; a different choice of ψ yields a different normal camera motion. The problem is overcome by working with the images, in each of which there is a well defined metric. The method is to examine the images 'near' to the image $\mathbf{p}$ taken from $\mathbf{x}$ and to select the best one for breaking the ambiguity. It is shown that to first order, the images near $\mathbf{p}$ define a tangent space which splits into an orthogonal sum $W_p \oplus N_p \oplus F_p$. The subspace W_p corresponds to images which preserve the three way ambiguity, N_p to images which break the ambiguity and F_p to impossible images, in that they cannot be obtained by motions of the camera.

1.1 Notation

The map $\pi_{\mathbf{x}} : \mathbb{P}^3 \to \mathbb{P}^2$ is any projection with centre $\mathbf{x}$. Let $S \subset \mathbb{P}^3$ be a set of m points in three dimensional projective space $\mathbb{P}^3$. Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of camera centres, and let $I = \{I_1, I_2, \ldots, I_n\}$ be a set of images of m points, in which each image point corresponds to a point of S. Then S, A is a reconstruction compatible with I if and only if there exists a projection $\pi_{\mathbf{a}_i} : \mathbb{P}^3 \to \mathbb{P}^2$ such that $\pi_{\mathbf{a}_i}(S)$ is collinear with I_i, $1 \le i \le n$. Let S', A' be a second reconstruction compatible with I. If there is a collineation $H : \mathbb{P}^3 \to \mathbb{P}^3$ such that $H(S) = S'$ and $H(A) = A'$, then S', A' is the same reconstruction as S, A. If no such collineation H exists then S, A and S', A' are different reconstructions, and reconstruction from I is said to be ambiguous.

Expressions of the form $\{S, \psi\}$ refer to a set S of six points in $\mathbb{P}^3$ contained in a quadric ψ.

2 Cremona Transformations

The proof that the three way ambiguity is maintained for all optical centres on ψ depends on the properties of a certain Cremona transformation ϕ of the plane which arises in the following way. Let $\{S_a, \psi_a\}$, $\{S_b, \psi_b\}$ be pairs such that for each point $\mathbf{x}$ on ψ_a there is a unique point $\alpha(\mathbf{x})$ on ψ_b with $\pi_{\mathbf{x}}(S_a)$ collinear to $\pi_{\alpha(\mathbf{x})}(S_b)$. After projecting ψ_a, ψ_b to planes, the map α corresponds to $\phi : \mathbb{P}^2 \to \mathbb{P}^2$.

Cremona transformations are highly constrained. This makes it easy to identify ϕ in the list of transformations given in [2]. Once this is done the main result follows quickly.

2.1 General properties

A Cremona transformation $\phi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ of order n is defined by an invertible map $\mathbf{x} \mapsto (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \phi_3(\mathbf{x}))^\top$ where the ϕ_i are homogeneous polynomials in $\mathbf{x}$ of degree n. The common zeros of the ϕ_i are, by definition, the base points or fundamental points $\mathbf{q}_i$, $1 \leq i \leq m$, of ϕ. At these points the value of ϕ is undefined.

The set of linear combinations of ϕ_1, ϕ_2, ϕ_3 is a two dimensional projective space L_ϕ. The condition that ϕ be invertible strongly constrains L_ϕ. For example, let f, g be general elements of L_ϕ. Then $\phi(f)$ and $\phi(g)$ are lines, and there exists a unique point $\mathbf{q}$ such that $\phi(\mathbf{q}) = \phi(f) \cap \phi(g)$. It follows that $\mathbf{q}$ is the only point of $f \cap g$ not coincident with a base point of ϕ.

The multiplicity t_i of ϕ at $\mathbf{q}_i$ is by definition the common multiplicity of f and g at $\mathbf{q}_i$, $1 \leq i \leq m$. An application of Bezout's theorem yields

$$n^2 = 1 + \sum_{i=1}^{m} t_i^2$$

If c is any plane curve, then the points of $f \cap c$ distinct from the base points of ϕ are known as free intersections.

A fundamental curve h of ϕ is by definition a curve such that $\phi(h)$ is a single point. This point is necessarily a fundamental point $\mathbf{r}$ of ϕ^{-1}. Each line l in the pencil of lines $P(\mathbf{r})$ centred at $\mathbf{r}$ maps to a curve $\phi^{-1}(l)$ which has exactly one free intersection with h. It follows that h is rational. The degree of h is equal to the multiplicity of ϕ^{-1} at $\mathbf{r}$.

Let n_i be the number of base points of ϕ of multiplicity t_i, $1 \leq i \leq m$. Then, in the notation of [10], ϕ is of type $C^n[t_1^{n_1}, \ldots, t_m^{n_m}]$. For example, the standard quadratic transformation is of type $C^2[1^3]$. It is noted that ϕ and ϕ^{-1} may have different types.

2.2 A particular Cremona transformation

The Cremona transformation ϕ relevant to reconstruction from images of six points is of type $C^{10}[6^1, 3^7]$. It is entry E''_{10} on the list given in §6 of [2]. The transformation ϕ^{-1} also has type $C^{10}[6^1, 3^7]$. The base points of ϕ are labelled $\mathbf{q}_i$, $0 \leq i \leq 7$, such that $\mathbf{q}_0$ has multiplicity 6 and the $\mathbf{q}_i$, $1 \leq i \leq 7$ each have multiplicity 3.

The fundamental curves of ϕ are described. For $1 \leq i \leq 7$, let c_i be the unique cubic curve that contains the $\mathbf{q}_j$, $0 \leq j \leq 7$, $j \neq i$ and that has a node at $\mathbf{q}_0$. The cubic c_i has no free intersection with a general curve of L_ϕ thus $\phi(c_i)$ is a single point, $\mathbf{r}_i$, which is necessarily a base point of ϕ^{-1}. The curve c_i meets a general line at three points thus ϕ^{-1} has multiplicity three at $\mathbf{r}_i$. Let c_0 be the unique curve of degree 6 with multiplicity 3 at $\mathbf{q}_0$ and multiplicity 2 at each $\mathbf{q}_i$, $1 \leq i \leq 7$. Then $\phi(c_0)$ is a base point $\mathbf{r}_0$ of multiplicity 6 for ϕ^{-1}. The points $\mathbf{r}_i$, $0 \leq i \leq 7$, are the fundamental points of ϕ^{-1}.

The existence of curves c_i, $0 \leq i \leq 7$, with the stated degrees and with the stated multiplicities at each $\mathbf{q}_j$, $0 \leq j \leq 7$ can be proved using quadratic transformations. For example, c_0 is obtained from a conic by applying in succession three quadratic transformations.

Let $l_i = \langle \mathbf{q}_0, \mathbf{q}_i \rangle$, $k_i = \langle \mathbf{r}_0, \mathbf{r}_i \rangle$, $1 \leq i \leq 7$. The line l_i has one free intersection with a general element of L_ϕ, thus $\phi(l_i)$ is a line. In addition, l_i has one free intersection with c_0 and one free intersection with c_i. It follows that $\phi(l_i)$ contains $\mathbf{r}_0$ and $\mathbf{r}_i$, hence $\phi(l_i) = k_i$, $1 \leq i \leq 7$.

The Cremona transformations of type $C^{10}[6^1, 3^7]$ have the following curious and useful property.

Theorem 1. *Let ϕ be a Cremona transformation of type $C^{10}[6^1, 3^7]$. Then the base points $\mathbf{q}_i$, $1 \leq i \leq 7$, are collinear with the basepoints $\mathbf{r}_i$, $1 \leq i \leq 7$.*

The proof of a similar result in §43 of [2] is readily adapted to include Theorem 1. In the applications of Theorem 1 made below, $\mathbf{q}_i$, $1 \leq i \leq 6$ and $\mathbf{r}_i$, $1 \leq i \leq 6$ are sets of points in different images and the theorem is invoked to show that the two images are collinear.

3 Three Way Ambiguity

In [9] Quan shows that there is a three way ambiguity in reconstruction from three images of six points. In this section it is shown that the three way ambiguity persists as new images are obtained, provided the optical centre of the camera remains on the unique quadric containing the scene points and the optical centres for the first three images.

The proof is in two parts: in §3.2 it is shown that each Cremona transformation of type $C^{10}[6^1, 3^7]$ gives rise to a three way ambiguity for all optical centres on a certain quadric. In the second part (§3.3) it is shown Quan's three reconstructions lead to a Cremona transformation of type $C^{10}[6^1, 3^7]$.

3.1 Definitions and a theorem

Definition 1. Let S_a, S_b be sets of six points in $\mathbb{P}^3$ and let ψ_a, ψ_b be non-singular quadrics such that $S_a \subset \psi_a$, $S_b \subset \psi_b$. Then $\{S_a, \psi_a\}$, $\{S_b, \psi_b\}$ are defined to be a linked pair if there exists a birational map $\alpha : \psi_a \rightarrow \psi_b$ such that $\pi_{\mathbf{x}}(S_a)$ is collinear with $\pi_{\alpha(\mathbf{x})}(S_b)$, and if in addition there is no collineation $\omega : \mathbb{P}^3 \to \mathbb{P}^3$ such that $\omega(S_a) = S_b$ and $\omega(\psi_a) = \omega(\psi_b)$.

Definition 2. A linked triple is defined to be a triple $\{S_a, \psi_a\}$, $\{S_b, \psi_b\}$, $\{S_c, \psi_c\}$ such that any two members of the triple are a linked pair.

When discussing linked pairs and linked triples the rational maps from ψ_a to ψ_b and from ψ_b to ψ_c are denoted by $\alpha : \psi_a \to \psi_b$ and $\beta : \psi_b \to \psi_c$.

Theorem 2. *Let $\{S_a, \psi_a\}$, $\{S_b, \psi_b\}$ be a linked pair such that no four points of S_a are coplanar and no four points of S_b are coplanar. Then the map $\alpha : \psi_a \to \psi_b$ is unique.*

Proof. Let $\mathbf{x}$ be a general point of ψ_a at which α is defined, and let $\mathbf{z}$ be a point of ψ_b such that $\pi_{\mathbf{z}}(S_b)$ is collinear with $\pi_{\alpha(\mathbf{x})}(S_b)$. Let five points $T \subset S_b$ be chosen, with the condition that if $\mathbf{z}$ is in S_b, then $T = S_b \setminus \{\mathbf{z}\}$. The point $\mathbf{x}$ is a general point of ψ_a thus no four points of $T \cup \{\alpha(\mathbf{x})\}$ are coplanar. Let c be the unique twisted cubic containing $T \cup \{\alpha(\mathbf{x})\}$. The point $\mathbf{z}$ is on c because $\pi_{\mathbf{z}}(T)$ is collinear with $\pi_{\alpha(\mathbf{x})}(T)$.

There are exactly two twisted cubics c_1, c_2 containing T and contained in ψ_b. In general, $\alpha(\mathbf{x})$ is not in $c_1 \cup c_2$, thus c is not contained in ψ_b. Hence $c \cap \psi_b$ contains exactly six points, which comprise $T \cup \{\alpha(\mathbf{x})\}$. It follows that $\mathbf{z} \in T \cup \{\alpha(\mathbf{x})\}$, hence $\mathbf{z} = \alpha(\mathbf{x})$ and α is unique. $\square$

3.2 Construction of a linked triple

The starting point for the construction is a Cremona transformation ϕ of type $C^{10}[6^1, 3^7]$. The following well known birational map from $\mathbb{P}^2$ to a non-singular quadric surface ψ in $\mathbb{P}^3$ is also required [10]. Let $\mathbf{u}$, $\mathbf{v}$ be distinct points of $\mathbb{P}^2$ and let f_i, $1 \leq i \leq 4$ be a basis for the three dimensional linear system of conics incident to $\mathbf{u}$ and $\mathbf{v}$. The map is

$$\mathbf{x} \mapsto (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}), f_4(\mathbf{x}))^\top \in \psi \qquad (1)$$

The line $\langle \mathbf{u}, \mathbf{v} \rangle$ maps to a single point $\mathbf{a}$. The inverse to (1) is a projection $\pi_{\mathbf{a}}$ with centre $\mathbf{a}$. The two generators of ψ through $\mathbf{a}$ project under $\pi_{\mathbf{a}}$ to $\mathbf{u}$ and $\mathbf{v}$ respectively.

Let ϕ have base points labelled as described in §2.2. Let $\psi_a \subset \mathbb{P}^3$ be the quadric obtained as the image of a map (1), in which $\mathbf{q}_0$, $\mathbf{q}_7$ play the roles of $\mathbf{u}$, $\mathbf{v}$, and let $\psi_b \subset \mathbb{P}^3$ be the quadric obtained with $\mathbf{r}_0$, $\mathbf{r}_7$ playing the roles of $\mathbf{u}$, $\mathbf{v}$. Let $\pi_{\mathbf{a}} : \psi_a \to \mathbb{P}^2$, $\pi_{\mathbf{b}} : \psi_b \to \mathbb{P}^2$ be the associated projections and let the points $\mathbf{x}_i$, $\mathbf{x}_i^{'}$ in $\mathbb{P}^3$ be defined by $\mathbf{x}_i = \pi_{\mathbf{a}}^{-1}(\mathbf{q}_i)$, $\mathbf{x}_i^{'} = \pi_{\mathbf{b}}^{-1}(\mathbf{r}_i)$, $1 \leq i \leq 6$. Let $\alpha : \psi_a \to \psi_b$ and the sets $S_a \subset \psi_a$, $S_b \subset \psi_b$ be defined by

$$\begin{aligned} \alpha &= \pi_{\mathbf{b}}^{-1} \circ \phi \circ \pi_{\mathbf{a}} \\ S_a &= \{\mathbf{x}_i, 1 \leq i \leq 6\} \\ S_b &= \{\mathbf{x}_i^{'}, 1 \leq i \leq 6\} \end{aligned} \qquad (2)$$

It is shown that α is not a collineation. For $1 \leq i \leq 7$, let c_i be the unique cubic plane curve which contains the $\mathbf{q}_j$, $0 \leq j \leq 7$, $j \neq i$, and which has a node at $\mathbf{q}_0$. Then $\pi_{\mathbf{a}}^{-1}(c_i)$ is the unique twisted cubic which is incident to the five points $S_a \setminus \{\mathbf{x}_i\}$ and which is incident twice to the generator $\pi_{\mathbf{a}}^{-1}(\mathbf{q}_0)$. The image $\alpha(\pi_{\mathbf{a}}^{-1}(c_i))$ is the single point $\mathbf{x}_i^{'}$, thus α is not the restriction of a collineation of $\mathbb{P}^3$. It follows from Theorem 2 that there is no collineation $\omega : \mathbb{P}^3 \to \mathbb{P}^3$ such that $\omega(S_a) = S_b$ and $\omega(\psi_a) = \psi_b$.

The image of S_a taken from $\mathbf{a}$ is collinear with the image of S_b taken from $\mathbf{b}$. This observation is extended in the following theorem.

Theorem 3. *Let $\mathbf{x}$ be a point of $\psi_a \setminus S_a$. Then the images $\pi_{\mathbf{x}}(S_a)$ and $\pi_{\alpha(\mathbf{x})}(S_b)$ are collinear.*

Proof. Let $\tilde{\mathbf{q}}_i = \pi_{\mathbf{x}}(\mathbf{x}_i)$, $1 \leq i \leq 6$, and let the two generators of ψ_a incident to $\mathbf{x}$ project to $\tilde{\mathbf{q}}_0$, $\tilde{\mathbf{q}}_7$, where the generator projecting to $\tilde{\mathbf{q}}_0$ is in the same family as the generator projecting to $\mathbf{q}_0$ at $\mathbf{a}$. The image of S_a, as viewed from $\mathbf{x}$, is transformed to the image of S_a, as viewed from $\mathbf{a}$ by applying a quadratic transformation θ with base points $\tilde{\mathbf{q}}_0$, $\tilde{\mathbf{q}}_7$, $\pi_{\mathbf{x}}(\mathbf{a})$. If $\mathbf{a}$ is on a generator of ψ_a incident to $\mathbf{x}$, then two of these base points are concident, but θ is still defined. Let $\tilde{\mathbf{r}}_i = \pi_{\alpha(\mathbf{x})}(\mathbf{x}_i^{'})$, $1 \leq i \leq 6$, and let the two generators of ψ_b at $\alpha(\mathbf{x})$ project to $\tilde{\mathbf{r}}_0$, $\tilde{\mathbf{r}}_7$, where the generator projecting to $\tilde{\mathbf{r}}_0$ is in the same family as the generator projecting to $\mathbf{r}_0$ at $\mathbf{b}$. The image of S_b, as viewed from $\alpha(\mathbf{x})$ is transformed to the image of S_b as viewed from $\mathbf{b}$ by applying the quadratic transformation θ' with base points $\tilde{\mathbf{r}}_0$, $\tilde{\mathbf{r}}_7$, $\pi_{\alpha(\mathbf{x})}(\mathbf{b})$.

Let $\xi : \mathbb{P}^2 \to \mathbb{P}^2$ be the Cremona transformation defined by $\xi = \theta^{'-1} \circ \phi \circ \theta$, as illustrated in the following commutative diagram.

$$\begin{tikzcd}
& \psi_a \arrow[r, "\alpha"] \arrow[d, "\pi_{\mathbf{x}}"] \arrow[ddl, swap, "\pi_{\mathbf{a}}"] & \psi_b \arrow[d, swap, "\pi_{\alpha(\mathbf{x})}"] \arrow[ddr, "\pi_{\mathbf{b}}"] & \\
& \mathbb{P}^2 \arrow[r, "\xi"] \arrow[dl, swap, "\theta"] & \mathbb{P}^2 \arrow[dr, "\theta'"] & \\
\mathbb{P}^2 \arrow[rrr, "\phi"] & & & \mathbb{P}^2
\end{tikzcd}$$

It is shown that ξ is of type $C^{10}[6^1, 3^7]$ by examining the curve $\xi(l)$, where l is a general line in the domain of ξ. The curve $\theta(l)$ is a conic incident to $\mathbf{q}_0$, $\mathbf{q}_7$, $\pi_{\mathbf{a}}(\mathbf{x})$ and $\phi(\theta(l))$ is a curve of degree 11 with multiplicity 3 at each $\mathbf{r}_i$, $1 \le i \le 6$, multiplicity 4 at $\mathbf{r}_7$ and multiplicity 7 at $\mathbf{r}_0$. The curve $\phi(\theta(l))$ contains $\pi_{\mathbf{b}}(\alpha(\mathbf{x}))$, because $\pi_{\mathbf{b}} \circ \alpha = \phi \circ \pi_{\mathbf{a}}$. A conic through $\pi_{\mathbf{b}}(\alpha(\mathbf{x}))$, $\mathbf{r}_0$, $\mathbf{r}_7$ meets $\phi(\theta(l))$ at 10 free intersections thus $\xi(l) = \theta'^{-1}(\phi(\theta(l)))$ has degree 10.

The curve $\xi(l)$ has multiplicity 6 at $\tilde{\mathbf{r}}_0$ and multiplicity 3 at each of the points $\tilde{\mathbf{r}}_i$, $1 \le i \le 7$. Thus ξ is of type $C^{10}[6^1, 3^7]$. An application of Theorem 1 shows that the points $\tilde{\mathbf{q}}_i$, $1 \le i \le 6$, are collinear with the points $\tilde{\mathbf{r}}_i$, $1 \le i \le 6$. □

Corollary. The pair $\{S_a, \psi_a\}$, $\{S_b, \psi_b\}$ is a linked pair.

A third reconstruction $\{S_c, \psi_c\}$ is obtained from ϕ by interchanging the roles of $\mathbf{q}_0$ and $\mathbf{q}_7$: let $\tilde{\phi}$ be the Cremona transformation with base points $\mathbf{q}_i$, $0 \le i \le 7$ in which $\mathbf{q}_7$ has multiplicity six and the remaining seven base points each have multiplicity three. The domain of $\tilde{\phi}$ yields the reconstruction $\{S_a, \psi_a\}$. The range of $\tilde{\phi}$ yields a new reconstruction $\{S_c, \psi_c\}$ not collinear with $\{S_a, \psi_a\}$ or with $\{S_b, \psi_b\}$. The reconstructions $\{S_a, \psi_a\}$, $\{S_b, \psi_b\}$, $\{S_c, \psi_c\}$ are a linked triple.

3.3 Construction of the Cremona transformation

It has been shown in §3.2 that each Cremona transformation of type $C^{10}[6^1, 3^7]$ gives rise to a linked triple. It remains to show that each reconstruction from three images of six points gives rise to a Cremona transformation of type $C^{10}[6^1, 3^7]$ and hence to a linked triple.

Theorem 4. *Let S_a, S_b, S_c be noncollinear sets of six object points compatible with three given images. Let the optical centres for S_a be $\mathbf{o}_1$, $\mathbf{o}_2$, $\mathbf{o}_3$. Then there is a Cremona transformation ϕ of type $C^{10}[6^1, 3^7]$ which yields a linked triple $\{S_a, \psi_a\}$, $\{S_b, \psi_b\}$, $\{S_c, \psi_c\}$ such that ψ_a contains $S_a \cup \{\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3\}$.*

Proof. Let ψ_a be the unique quadric containing $S_a \cup \{\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3\}$, let $\mathbf{a}$ be a general point of ψ_a and define base points $\mathbf{q}_i$, $0 \le i \le 7$ in $\mathbb{P}^2$ such that $\{\mathbf{q}_i, 1 \le i \le 6\} = \pi_{\mathbf{a}}(S_a)$, and $\mathbf{q}_0$, $\mathbf{q}_7$ are the projections of the two generators of ψ_a through $\mathbf{a}$. The base points $\mathbf{q}_i$, $0 \le i \le 7$ define a unique Cremona transformation ϕ of type $C^{10}[6^1, 3^7]$ for which $\mathbf{q}_0$ has multiplicity 6.

The transformation ϕ yields a linked triple $\{S_a, \psi_a\}$, $\{S_{b'}, \psi_{b'}\}$, $\{S_{c'}, \psi_{c'}\}$. In [9] it is shown that S_a, S_b, S_c are the only sets of scene points (up to a collineation) which are compatible with the three images, thus $S_{b'} = S_b$, $S_{c'} = S_c$. □

It can be shown that a linked triple $\{S_a, \psi_a\}$, $\{S_b, \psi_b\}$, $\{S_c, \psi_c\}$ is real if and only if $\{S_a, \psi_a\}$ is real and ψ_a is a rectangular hyperboloid. Here, $\{S_a, \psi_a\}$ real means that the points in S_a have coordinates in $\mathbb{R}$ and ψ_a is defined over $\mathbb{R}$.

4 Breaking the Ambiguity

The three way ambiguity can be resolved by a further motion of the camera, but the question arises: which motion is best? As noted in §1, the scene geometry, including the quadric ψ, is defined only up to a collineation, thus there is no metric for comparing different motions in $\mathbb{P}^3$. For this reason, camera motions are described by their effects on the image. Each image has a well defined metric which can be used to make comparisons between different image perturbations. The idea of focusing on images rather than on the three dimensional world is suggested by [13].

In this section collineations are applied to each image to move the first four points to the positions they hold in the first image. The aim is to reduce the number of calculations, but this is at the cost of distorting the image metric. If the optical centres are close and the changes in camera calibration are small then the distortions of the metric should be small.

4.1 The space of all images

Let three images of six points S be obtained with optical centres $\mathbf{o}_1$, $\mathbf{o}_2$, $\mathbf{o}_3$. Collineations are applied to the second and third images to bring the first four points of each image into alignment with the first four points of $\mathbf{o}_1$. With the first four points fixed, each image has just four degrees of freedom, namely the inhomogeneous coordinates of the remaining two points. The image can thus be represented by a point in $\mathbb{R}^4$.

Let $\mathbf{p}(\mathbf{x})$ be the point of $\mathbb{R}^4$ corresponding to the image obtained with the optical centre $\mathbf{x}$ in $\mathbb{P}^3$. The range of the map $\mathbf{x} \mapsto \mathbf{p}(\mathbf{x})$ is a three dimensional variety $V \subset \mathbb{R}^4$. It follows immediately that 'most' points in $\mathbb{R}^4$ do not correspond to physically realisable images of S, in that they cannot be produced by a suitable choice of camera position and calibration. Let ψ be the quadric containing $S \cup \{\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3\}$, let W be the image of ψ in $\mathbb{R}^4$ under the map $\mathbf{x} \mapsto \mathbf{p}(\mathbf{x})$, and let $\mathbf{p} = \mathbf{p}(\mathbf{o}_3)$.

The tangent space T_p to $\mathbb{R}^4$ at $\mathbf{p}$ is four dimensional. In the general case, when V, W are both smooth at $\mathbf{p}$, T_p splits into an orthogonal sum,

$$T_p = W_p \oplus N_p \oplus F_p \tag{3}$$

where W_p is the two dimensional tangent space to W at $\mathbf{p}$, N_p is the orthogonal complement to W_p in the tangent space of V at $\mathbf{p}$, and F_p is the orthogonal complement of $W_p \oplus N_p$ in T_p. The spaces N_p, F_p both have dimension one.

The decomposition (3) gives a complete first order description of the effects of a small camera displacement from $\mathbf{o}_3$. Let a fourth image be obtained using an optical centre close to $\mathbf{o}_3$, and let the correspond point in $\mathbb{R}^4$ be $\mathbf{p}+\mathbf{t}$. If $\mathbf{t}$ is close to a vector in W_p, then the new image will do little to resolve the ambiguity. If $\mathbf{t}$ is close to a vector in N_p then the ambiguity will be resolved most easily. Finally, a significant component of $\mathbf{t}$ in the direction F_p indicates an error.

4.2 Example

A computer simulation was carried out in Mathematica [14]. Rational arithmetic was used, as far as possible, to avoid floating point errors. The optical centres were $\mathbf{o}_1 = (0,0,0,1)^\top$, $\mathbf{o}_2 = (1/4,0,0,1)^\top$, $\mathbf{o}_3 = (1/2,0,0,1)^\top$. The six object points were chosen at random in the unit cube centred on $(0,0,8,1)^\top$. The images were formed by projection onto the plane $x_3 = 5x_4$. The quadric ψ has an equation $\mathbf{x}^\top M\mathbf{x} = 0$ where M is a symmetric 4×4 matrix. The matrix M was obtained by solving the nine simultaneous linear equations $\mathbf{o}_i^\top M\mathbf{o}_i = 0$, $1 \le i \le 3$, and $\mathbf{x}^\top M\mathbf{x} = 0$, $\mathbf{x} \in S$.

The point $\mathbf{p}(\mathbf{x})$ in $\mathbb{R}^4$ corresponding to an optical centre $\mathbf{x} = (x_1, x_2, x_3, x_4)^\top$ in $\mathbb{P}^3$ was found in closed form. The formula for $\mathbf{p}(\mathbf{x})$ is large, and the details are unilluminating, thus it is omitted from this description.

The spaces W_p, N_p, F_p were obtained as follows. Let Π be the tangent plane to ψ at $\mathbf{o}_3$ and let $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ be points of $\mathbb{P}^3$ chosen such that for $\lambda \neq 0$, $\mathbf{o}_3 + \lambda\mathbf{m}_1$, $\mathbf{o}_3 + \lambda\mathbf{m}_2$ are in Π and $\mathbf{o}_3 + \lambda\mathbf{m}_3$ is not in Π. The tangent vectors $\mathbf{n}'_1$, $\mathbf{n}'_2$, $\mathbf{n}'_3$ were defined in $\mathbb{R}^4$ by

$$\mathbf{n}'_i = \frac{\partial}{\partial\lambda}\mathbf{p}(\mathbf{o}_3 + \lambda\mathbf{m}_i)\bigg|_{\lambda=0} \qquad (i = 1,2,3)$$

Then $\mathbf{n}'_1$, $\mathbf{n}'_2$ span W_p, and $\mathbf{n}'_1$, $\mathbf{n}'_2$, $\mathbf{n}'_3$ span $W_p \oplus N_p$.

Let $\mathbf{n}_4$ be a unit vector normal to $\langle \mathbf{n}'_1, \mathbf{n}'_2, \mathbf{n}'_3\rangle$, let $\mathbf{n}_3$ be a unit vector normal to $\langle \mathbf{n}'_1, \mathbf{n}'_2, \mathbf{n}_4\rangle$ and let $\{\mathbf{n}_1, \mathbf{n}_2\}$ be an orthonormal base of $\langle \mathbf{n}'_1, \mathbf{n}'_2\rangle$. It follows that $\{\mathbf{n}_1, \mathbf{n}_2\}$ is an orthonormal base of W_3, $\mathbf{n}_3$ spans N_3 and $\mathbf{n}_4$ spans F_4. The directions of the vectors $\mathbf{n}_i$, $1 \le i \le 4$, were computed rationally but real arithmetic was used to scale them to unit length.

In order to resolve the ambiguity most effectively it is necessary to move the camera such that the resulting image is close to the line $\lambda \mapsto \mathbf{p} + \lambda\mathbf{n}_3$. On the other hand, if the new image is close to the plane $(\lambda_1, \lambda_2) \mapsto \mathbf{p} + \lambda_1\mathbf{n}_1 + \lambda_2\mathbf{n}_3$, then the ambiguity cannot be resolved.

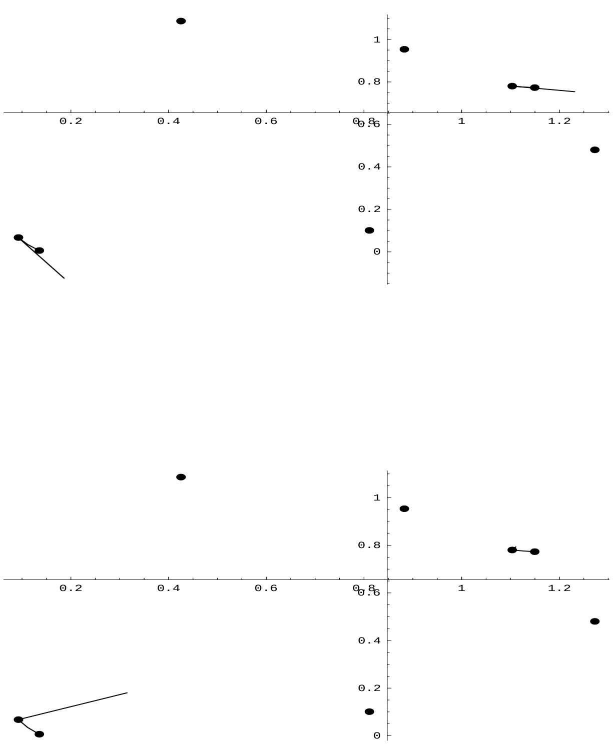

Figure 1: Image perturbations which preserve the ambiguity.

The results are shown in Figures 1 and 2. In each figure the four isolated points are the fixed points in each image. The positions of the fifth and sixth points in successive images are shown by the polygonal arcs. The bullet at the beginning of the polygonal arc is the position of the point in the first image, $\pi_{\mathbf{o}_1}(S)$. The second bullet is the position of the corresponding point in the third image, $\pi_{\mathbf{o}_3}(S)$. The line at the end of the polygonal arc shows a perturbation of the image obtained using a vector chosen from the tangent space T_p at $\mathbf{p}(\mathbf{o}_3)$ in $\mathbb{R}^4$.

The two images in Figure 1 show orthogonal perturbations drawn from W_p. To first order, these perturbations fail to resolve the three way ambiguity. The top image in Figure 2 shows a perturbation spanning N_p. This is the best choice to resolve the three way ambiguity. The bottom image in Figure 2 shows a physically impossible perturbation of the image.

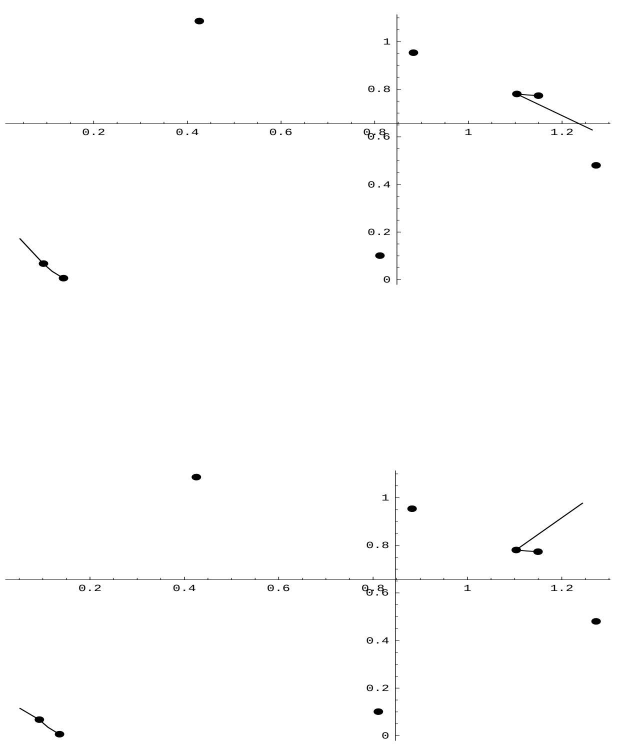

Figure 2, top: Best perturbation for breaking the ambiguity; bottom: impossible perturbation.

5 Conclusion

It has been shown that reconstruction from a sequence of images of six points is subject to a three way ambiguity, which is preserved as long as the optical centre of the camera remains on a fixed rectangular hyperboloid containing the six points. This unexpected result increases our understanding of the geometry underlying scene reconstruction from image sequences. A method has been described for identifying the best 'next image' for resolving the ambiguity. The method may be adapted for planning camera trajectories in active vision systems.

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