

# Multiple View Geometry of Non-planar Algebraic Curves\*

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## Abstract

*We introduce a number of new results in the context of multi-view geometry from general algebraic curves. We start with the derivation of the extended Kruppa's equations which are responsible for describing the epipolar constraint of two projections of a general (non-planar) algebraic curve. As part of the derivation of those constraints we address the issue of dimension analysis and as a result establish the minimal number of algebraic curves required for a solution of the epipolar geometry as a function of their degree and genus.*

*We then establish new results on the reconstruction of general algebraic curves from multiple views. We address three different representations of curves: (i) the regular point representation for which we show that the reconstruction from two views of a curve of degree  $d$  admits two solutions, one of degree  $d$  and the other of degree  $d(d - 1)$ , (ii) the dual space representation (tangents) for which we derive a lower bound for the number of views necessary for reconstruction as a function of the curve degree and genus, and (iii) a new representation (to computer vision) based on the set of lines meeting the curve which does not require any curve fitting in image space, for which we also derive lower bounds for the number of views necessary for reconstruction as a function of the curve degree alone.*

## 1. Introduction

A large body of research has been devoted to the problem of analyzing the 3D structure of a scene from multiple views. The multi-view theory is by now well understood when the scene consists of point and line features — a sum-

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\*Part of this work was done while J.Y.K was at the Hebrew University pursuing his doctoral studies.

mary of the past decade of work in this area can be found in [13] and references to earlier work in [6].

The theory is somewhat fragmented when it comes to curve features, especially non-planar algebraic curves of general degree. Given known projection matrices [23, 18, 19] show how to recover the 3D position of a conic section from two and three views, and [24] show how to recover the homography matrix of the conic plane, and [11, 25] show how to recover a quadric surface from projections of its occluding conics.

Reconstruction of higher-order curves were addressed in [16, 3, 21, 22]. In [3] the matching curves are represented parametrically where the goal is to find a re-parameterization of each matching curve such that in the new parameterization the points traced on each curve are matching points. The optimization is over a discrete parameterization, thus, for a planar algebraic curve of degree  $n$ , which is represented by  $\frac{1}{2}n(n + 3)$  points, one would need  $n(n + 3)$  minimal number of parameters to solve for in a non-linear bundle adjustment machinery — with some prior knowledge of a good initial guess. In [21, 22] the reconstruction is done under infinitesimal motion assumption with the computation of spatio-temporal derivatives that minimize a set of non-linear equations at many different points along the curve. In [16] only planar algebraic curves were considered.

The literature is somewhat sparse on the problem of recovering the camera projection matrices from matching projections of algebraic curves. For example, [15, 16] show how to recover the fundamental matrix from matching conics with the result that 4 matching conics are minimally necessary for a unique solution. [16] generalize this result to higher order curves, but consider only planar curves.

In this paper we address the general issue of multi-view geometry of (non-planar) algebraic curves starting with the derivation of the *extended Kruppa's equations* which are re-

sponsible for describing the epipolar constraint of two projections of a general (non-planar) algebraic curve. As part of the derivation of those constraints we address the issue of dimension analysis and as a result establish the minimal number of algebraic curves required for a solution of the epipolar geometry as a function of their degree and genus.

On the reconstruction front, we address three different representations of curves: (i) the regular point representation for which we show that the reconstruction from two views of a curve of degree  $d$  admits two solutions, one of degree  $d$  and the other of degree  $d(d - 1)$ , (ii) dual space representation (image measurements are tangent lines) for which we derive a formula for the minimal number of views necessary for reconstruction as a function of the curve degree and genus, and (iii) a new representation (with regard to computer vision) based on the set of lines meeting the curve which does not require any curve fitting in image space, for which we also derive formulas for the minimal number of views necessary for reconstruction as a function of curve degree alone.

## 2. Recovering the epipolar geometry from curve correspondences

An algebraic relation between matching algebraic curves can be regarded as an extension of Kruppa's equations. In their original form, these equations have been introduced to compute the camera-intrinsic parameters from the projection of the absolute conic onto the two image planes [20]. However it is obvious that they still hold if one replaces the absolute conic by any conic, or even any planar algebraic curve, which does not meet any of the camera centers [16]. We will show next that a generalization for arbitrary algebraic (non-planar) curves is possible and is a step toward the recovery of epipolar geometry from matching curves.

Let  $X$  be a smooth irreducible (non-planar) curve in  $\mathbb{P}^3$ , whose degree is  $d \geq 2$ . Let  $\mathbf{M}$  be the camera matrix,  $\mathbf{O}$  the camera center,  $\mathcal{R}$  the retinal plane and  $Y$  the image curve. Here we mention a short list of well known facts (see [14, 12, 5, 4]):

1. We recall that a singularity of a planar curve is simply a point where the curve admits more than one tangent. The curve  $Y$  will always contain singularities.
2. For a generic position of the camera center, the only singularities of  $Y$  will be nodes, that is points with two distinct tangents.
3. We define the *class* of the planar curve to be the degree of its dual curve. Let  $m$  be the class of  $Y$ . It is a well-known fact that  $m$  is constant for a generic position of the camera center.

4. For a generic position of  $\mathbf{O}$ ,  $Y$  will have same degree and genus as  $X$

Note it is possible to give a formula for  $m$  as a function of the degree  $d$  and the genus  $g$  of  $X$ . According to Plücker formula, we have:

$$m = d(d - 1) - 2(\#\text{nodes}),$$

$$g = \frac{(d - 1)(d - 2)}{2} - (\#\text{nodes}),$$

where  $\#\text{nodes}$  denotes the number of nodes of  $Y$  (note that this includes complex nodes). Hence the genus, the degree and the class are related by:

$$m = 2d + 2g - 2.$$

### 2.1. Extended Kruppa's equations

We are ready now to investigate the recovery of the epipolar geometry from matching curves. Let  $\mathbf{M}_i, i = 1, 2$ , be the camera matrices. Let  $\mathbf{F}$  and  $\mathbf{e}_1$  be the fundamental matrix and the first epipole,  $\mathbf{F}\mathbf{e}_1 = 0$ . We will need to consider the two following mappings:  $\gamma : \mathbf{p} \rightarrow \mathbf{e}_1 \wedge \mathbf{p}$  and  $\xi : \mathbf{p} \rightarrow \mathbf{F}\mathbf{p}$ . Both are defined on the first image plane;  $\gamma$  associates a point to its epipolar line in the first image, while  $\xi$  sends it to its epipolar line in the second image.

Let  $Y_1$  and  $Y_2$  be the image curves (projections of  $X$  onto the image planes). Let  $f_1$  and  $f_2$  be the polynomials that represent respectively  $Y_1$  and  $Y_2$ . Let  $Y_1^*$  and  $Y_2^*$  denote the dual curves, whose polynomials are respectively  $\phi_1$  and  $\phi_2$ . Roughly speaking, the extended Kruppa's equations state that the sets of epipolar lines tangent to the curve in each image are projectively related. A similar observation has been made in [2] for epipolar lines tangent to apparent contours of objects, but it was used within an optimization scheme. Here we are looking for closed-form solutions, where no initial knowledge of the answer is required. In order to develop such a closed-form solution for the computation of the epipolar geometry, we need a more quantitative approach, which is given by the following theorem:

#### Theorem 1 Extended Kruppa's equations

*For a generic position of the camera centers with respect to the curve in space, there exists a non-zero scalar  $\lambda$ , such that for all points  $\mathbf{p}$  in the first image, the following equality holds:*

$$\phi_2(\xi(\mathbf{p})) = \lambda\phi_1(\gamma(\mathbf{p})) \quad (1)$$

Observe that if  $X$  is a conic and  $\mathbf{C}_1$  and  $\mathbf{C}_2$  the matrices that respectively represent  $Y_1$  and  $Y_2$ , the extended Kruppa's equations reduce to the classical Kruppa's equations, that is:  $[\mathbf{e}_1]_x^T \mathbf{C}_1^* [\mathbf{e}_1]_x \cong \mathbf{F}^T \mathbf{C}_2^* \mathbf{F}$ , where  $\mathbf{C}_1^*$  and  $\mathbf{C}_2^*$  are the adjoint matrices of  $\mathbf{C}_1$  and  $\mathbf{C}_2$ .

*Proof:* Let  $\epsilon_i$  be the set of epipolar lines tangent to the curve in image  $i$ . We start by proving the following lemma.

**Lemma 1** *The two sets  $\epsilon_1$  and  $\epsilon_2$  are projectively equivalent. Furthermore for each corresponding pair of epipolar lines,  $(\mathbf{l}, \mathbf{l}') \in \epsilon_1 \times \epsilon_2$ , the multiplicity of  $\mathbf{l}$  and  $\mathbf{l}'$  as points of the dual curves  $Y_1^*$  and  $Y_2^*$  are the same.*

*Proof:* Consider the three following pencils:

1.  $\sigma(\mathbf{L}) \cong \mathbb{P}^1$ , the pencils of planes containing the baseline, generated by the camera centers,
2.  $\sigma(\mathbf{e}_1) \cong \mathbb{P}^1$ , the pencil of epipolar lines through the first epipole,
3.  $\sigma(\mathbf{e}_2) \cong \mathbb{P}^1$ , the pencil of epipolar lines through the second epipole.

Thus we have  $\epsilon_i \subset \sigma(\mathbf{e}_i)$ . Moreover if  $E$  is the set of plane in  $\sigma(\mathbf{L})$  tangent to the curve in space, there exist a one-to-one mapping from  $E$  to each  $\epsilon_i$ . This mapping also leaves the multiplicities unchanged. This completes the lemma. ■

This lemma implies that both side of the equation 1 define the same algebraic set, that the union of epipolar lines through  $\mathbf{e}_1$  tangent to  $Y_1$ . Since  $\phi_1$  and  $\phi_2$ , in the generic case, have same degree (as stated in 2), each side of equation 1 can be factorized into linear factors, satisfying the following:

$$\begin{aligned}\phi_1(\gamma(x, y, z)) &= \prod_i (\alpha_{1i}x + \alpha_{2i}y + \alpha_{3i}z)^{a_i} \\ \phi_2(\xi(x, y, z)) &= \prod_i \lambda_i (\alpha_{1i}x + \alpha_{2i}y + \alpha_{3i}z)^{b_i},\end{aligned}$$

where  $\sum_i a_i = \sum_j b_j = m$ . By the previous lemma, we must also have  $a_i = b_i$  for  $i$ . ■

By eliminating the scalar  $\lambda$  from the extended Kruppa's equations (1) we obtain a set of bi-homogeneous equations in  $\mathbf{F}$  and  $\mathbf{e}_1$ . Hence they define a variety in  $\mathbb{P}^2 \times \mathbb{P}^8$ . This gives rise to an important question. How many of those equations are algebraically independent, or in other words what is the dimension of the set of solutions? This is the issue of the next section.

## 2.2. Dimension of the set of solutions

Let  $\{E_i(\mathbf{F}, \mathbf{e}_1)\}_i$  be the set of bi-homogeneous equations on  $\mathbf{F}$  and  $\mathbf{e}_1$ , extracted from the extended Kruppa's equations (1). Our first concern is to determine whether all solutions of equation (1) are admissible, that is whether they satisfy the usual constraint  $\mathbf{F}\mathbf{e}_1 = 0$ . Indeed the following statement can be proven [1]:

**Proposition 1** *As long as there are at least 2 distinct lines through  $\mathbf{e}_1$  tangent to  $Y_1$ , equation (1) implies that  $\text{rank } \mathbf{F} = 2$  and  $\mathbf{F}\mathbf{e}_1 = 0$ .*

As a result, in a generic situation every solution of  $\{E_i(\mathbf{F}, \mathbf{e}_1)\}_i$  is admissible. Let  $V$  be the subvariety of  $\mathbb{P}^2 \times \mathbb{P}^8 \times \mathbb{P}^2$  defined by the equations  $\{E_i(\mathbf{F}, \mathbf{e}_1)\}_i$  together with  $\mathbf{F}\mathbf{e}_1 = 0$  and  $\mathbf{e}_2^T \mathbf{F} = 0^T$ , where  $\mathbf{e}_2$  is the second epipole. We have the two following results, which are proven in [1]. The first of them gives a lower bound of the dimension of  $V$  as a function of  $m$ , whereas the second gives a sufficient condition for a  $V$  to be a finite set.

**Proposition 2** *If  $V$  is non-empty, the dimension of  $V$  is at least  $7 - m$ .*

**Proposition 3** *For a generic position of the camera centers, the variety  $V$  will be discrete if, for any point  $(\mathbf{e}_1, \mathbf{F}, \mathbf{e}_2) \in V$ , the union of  $\mathbf{L}$  and the points  $\mathbf{Q}_\alpha(\mathbf{e}_1, \mathbf{e}_2)$  is not contained in any quadric surface.*

Observe that these two results are consistent, since there always exist a quadric surface containing a given line and six given points. However in general there is no quadric containing a given line and seven given points. Therefore we can conclude with the following theorem.

**Theorem 2** *For a generic position of the camera centers, the extended Kruppa's equations define the epipolar geometry up to a finite-fold ambiguity if and only if  $m \geq 7$ .*

Since different curves in generic position give rise to independent equations, this result means that the sum of the classes of the image curves must be at least 7 for  $V$  to be a finite set. Observe that this result is consistent with the fact that four conics ( $m = 2$  for each conic) in general position are sufficient to compute the fundamental matrix, as shown in [15, 16]. Now we proceed to translate the result in terms of the geometric properties of  $X$  directly using the degree and the genus of  $X$ , related to  $m$  by the following relation:  $m = 2d + 2g - 2$ . Here are some examples for sets of curves that allow the recovery of the fundamental matrix:

1. Four conics ( $d = 2, g = 0$ ) in general position.
2. Two rational cubics ( $d = 3, g = 0$ ) in general position.
3. A rational cubic and two conics in general position.
4. Two elliptic cubics ( $d = 3, g = 1$ ) in general position (see also [16]).
5. A general rational quartic ( $d = 4, g = 0$ ), and a general elliptic quartic ( $d = 4, g = 1$ ).

## 3. 3D Reconstruction

We turn our attention to the problem of reconstructing an algebraic curve from two or more views, given known camera matrices (epipolar geometries are known). The basic

idea is to intersect together the cones defined by the camera centers and the image curves. However this intersection can be computed in three different spaces, giving rise to different algorithms and applications. Given the representation in one of those spaces, it is possible to compute the two other representations [1].

We shall mention that in [10] a scheme is proposed to reconstruct an algebraic curve from a single view by blowing-up the projection. This approach results in a spatial curve defined up to an unknown projective transformation. In fact the only computation this reconstruction allows is the recovery of the projective properties of the curve. Moreover this reconstruction is valid for irreducible curves only. However reconstructing from two projections not only gives the projective properties of the curve, but also the relative depth of it with respect to others objects in the scene and furthermore the relative position between irreducible components.

### 3.1. Reconstruction in Point Space

Let the camera projection matrices be  $[\mathbf{I}; \mathbf{0}]$  and  $[\mathbf{S}; \mathbf{e}_2]$ , where  $\mathbf{S} = -\frac{[\mathbf{e}_2]_{\times}}{\|\mathbf{e}_2\|^2} \mathbf{F}$ , see [17]. Hence the two cones defined by the image curves and the camera centers are given by:  $\Delta_1(\mathbf{P}) = f_1([\mathbf{I}; \mathbf{0}]\mathbf{P})$  and  $\Delta_2(\mathbf{P}) = f_2([\mathbf{S}; \mathbf{e}_2]\mathbf{P})$ . The reconstruction is defined as the curve whose equations are  $\Delta_1 = 0$  and  $\Delta_2 = 0$ . The irreducible components of the intersection (the separate curves) have the following degrees:

**Theorem 3** *For a generic position of the camera centers, that is when no epipolar plane is tangent twice to the curve  $X$ , the curve defined by  $\{\Delta_1 = 0, \Delta_2 = 0\}$  has two irreducible components. One has degree  $d$  and is the actual solution of the reconstruction. The other one has degree  $d(d-1)$ .*

We skip the proof because of its technical content. The reader could find it in [1]. This result provides an algorithm to find the right solution for the reconstruction in a generic configuration, except in the case of conics, where the two components of the reconstruction are both admissible.

### 3.2. Reconstruction in the Dual Space

Let  $X^*$  be the dual variety of  $X$ , that is, the set of planes tangent to  $X$ . Since  $X$  is supposed not to be a line, the dual variety  $X^*$  must be a hypersurface of the dual space [12]. Hence let  $\Upsilon$  be a minimal degree polynomial that represents  $X^*$ . Our first concern is to determine the degree of  $\Upsilon$ .

**Proposition 4** *The degree of  $\Upsilon$  is  $m$ , that is, the common degree of the dual image curves.*

*Proof:* Since  $X^*$  is a hypersurface of  $\mathbb{P}^{3*}$ , its degree is the number of points where a generic line in  $\mathbb{P}^{3*}$  meets  $X^*$ .

By duality it is the number of planes in a generic pencil that are tangent to  $X$ . Hence it is the degree of the dual image curve. Another way to express the same fact is the observation that the dual image curve is the intersection of  $X^*$  with a generic plane in  $\mathbb{P}^{3*}$ . Note that this provides a new proof that the degree of the dual image curve is constant for a generic position of the camera center. ■

For the reconstruction of  $X^*$  from multiple view, we will need to consider the mapping from a line  $\mathbf{l}$  of the image plane to the plane that it defines with the camera center. Let  $\mu : \mathbf{l} \mapsto \mathbf{M}^T \mathbf{l}$  denote this mapping [7]. There exists a link involving  $\Upsilon$ ,  $\mu$  and  $\phi$ , the polynomial of the dual image curve:  $\Upsilon(\mu(\mathbf{l})) = 0$  whenever  $\phi(\mathbf{l}) = 0$ . Since these two polynomials have the same degree (because  $\mu$  is linear) and  $\phi$  is irreducible, there exist a scalar  $\lambda$  such that

$$\Upsilon(\mu(\mathbf{l})) = \lambda\phi(\mathbf{l}),$$

for all lines  $\mathbf{l} \in \mathbb{P}^{2*}$ . Eliminating  $\lambda$ , we get  $\frac{(m+2)(m+1)}{2} - 1$  linear equations on  $\Upsilon$ . Since the number of coefficients in  $\Upsilon$  is  $\frac{(m+3)(m+2)(m+1)}{6}$ , we can state the following result:

**Proposition 5** *The reconstruction in the dual space can be done linearly using at least  $k \geq \frac{m^2+6m+11}{3(m+3)}$  views.*

The lower bounds on the number of views  $k$  for few examples are given below:

1. for a conic locus,  $k \geq 2$ ,
2. for a rational cubic,  $k \geq 3$ ,
3. for an elliptic cubic,  $k \geq 4$ ,
4. for a rational quartic,  $k \geq 4$ ,
5. for an elliptic quartic,  $k \geq 4$ .

Moreover it is worth noting that the fitting of the dual image curve is not necessary. It is sufficient to extract tangents to the image curves at distinct points. Each tangent  $\mathbf{l}$  contributes to one linear equation on  $\Upsilon$ :  $\Upsilon(\mu(\mathbf{l})) = 0$ . However one cannot obtain more than  $\frac{(m+2)(m+1)}{2} - 1$  linearly independent equations per view.

### 3.3. Reconstruction in $\mathbb{G}(1, 3)$

As a third representation of the curve  $X$ , we consider the set of lines meeting  $X$ . This defines completely the curve  $X$ , as shown in [12]. As we shall see, we will pay the price of requiring extra views for reconstruction but will gain from the fact that we can use the image points directly without the need to perform curve fitting.

A line can either be represented by a pair of planes  $(\mathbf{H}_1, \mathbf{H}_2) \in \mathbb{P}^{3*} \times \mathbb{P}^{3*}$  that contain it or by its Plücker coordinates in the Grassmannian  $\mathbb{G}(1, 3)$ . For the purpose of

computation with curves, we found that the representation by Plücker coordinates is more convenient.

Hence the curve  $X$  of degree  $d$  can be represented by a homogeneous polynomial  $\Gamma$ , called in that context the Chow polynomial of  $X$ , of degree  $d$ , that defines a hypersurface in  $\mathbb{G}(1, 3)$ . Note that  $\Gamma$  is not uniquely defined. Two such polynomials must differ by a multiple of the quadratic equation defining  $\mathbb{G}(1, 3)$  as a subvariety of  $\mathbb{P}^5$ . However picking one representative of this equivalence class is sufficient to reconstruct entirely without any ambiguity the curve  $X$  [12]. The number of coefficients in  $\Gamma$  is  $\binom{d+5}{d}$ . However since it is defined modulo the quadratic equation defining  $\mathbb{G}(1, 3)$ , it is sufficient to provide  $\binom{d+5}{d} - \binom{d-2+5}{d-2}$  independent linear conditions on its coefficients to determine one instance of  $\Gamma$ .

Let  $f$  be the polynomial defining the image curve,  $Y$ . Consider the mapping that associates to an image point its optical ray:  $\nu : \mathbf{p} \mapsto \widehat{\mathbf{M}}\mathbf{p}$ , where  $\widehat{\mathbf{M}}$  is a  $3 \times 6$  matrix, whose entries are polynomial functions of  $\mathbf{M}$  [7]. Hence the polynomial  $\Gamma(\nu(\mathbf{p}))$  vanishes whenever  $f(\mathbf{p})$  does. Since they have the same degree and  $f$  is irreducible, there exists a scalar  $\lambda$  such as for every point  $\mathbf{p} \in \mathbb{P}^2$ , we have:

$$\Gamma(\nu(\mathbf{p})) = \lambda f(\mathbf{p}).$$

This yields  $\binom{d+2}{d} - 1$  linear equations on  $\Gamma$ .

Hence a similar statement to that in Proposition 5 can be made:

**Proposition 6** *The reconstruction in  $\mathbb{G}(1, 3)$  can be done linearly using at least  $k \geq \frac{1}{6} \frac{d^3 + 5d^2 + 8d + 4}{d}$  views.*

For some examples, below are the minimal number of views for a linear reconstruction of the curve in  $\mathbb{G}(1, 3)$ :

1. for a conic locus,  $k \geq 4$ ,
2. for a cubic,  $k \geq 6$ ,
3. for a quartic,  $k \geq 8$ .

As in the case of reconstruction in the dual space, it is not necessary to explicitly compute  $f$ . It is enough to pick points on the image curve. Each point yields a linear equation on  $\Gamma$ :  $\Gamma(\nu(\mathbf{p})) = 0$ . However for each view, one cannot extract more than  $\frac{1}{2}d^2 + \frac{3}{2}d$  independent linear equations.

## 4. Experiments

We start with a synthetic experiment followed later by a real image one. Consider the curve  $X$ , drawn in figure 1, defined by the following equations:

$$F_1(x, y, z, t) = x^2 + y^2 - t^2$$

$$F_2(x, y, z, t) = xt - (z - 10t)^2$$

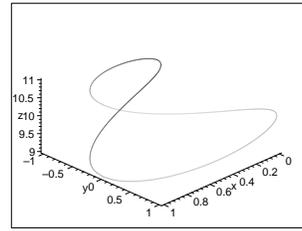


Figure 1. A spatial quartic



Figure 2. An electric thread.

The curve  $X$  is smooth and irreducible, and has degree 4 and genus 1. We define two camera matrices:

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & -1 & 0 & -10 \end{bmatrix} \quad \mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -10 \end{bmatrix}$$

The reconstruction of the curve from the two projections has been made in the point space, using **FGb**<sup>1</sup>, a powerful software tool for Gröbner basis computation [8, 9]. As expected there are two irreducible components. One has degree 4 and is the original curve, while the second has degree 12.

For the next experiment, we consider seven images of an electric wire — one of the views is shown in figure 2. In each image, we perform segmentation and thinning of the image curve. Hence for each of the images, we extracted a set of points lying on the thread. No fitting is performed in the image space. For each image, the camera matrix is calculated using the calibration pattern. Then we proceeded to compute the Chow polynomial  $\Gamma$  of the curve in space. The curve  $X$  has degree 3. Once  $\Gamma$  is computed, a reprojection is easily performed, as shown in figure 3.

## 5. Conclusion

In this paper we have focused on general (non-planar) algebraic curves as the building blocks from which the camera geometries are to be recovered and as the scene building blocks for the purpose of reconstruction from multiple views. The new results derived in this paper include:

<sup>1</sup>Logiciel conçu et réalisé au laboratoire LIP6 de l'université Pierre et Marie CURIE.



**Figure 3. Reprojection on a new image.**

1. Extended Kruppa's equations for the recovery of epipolar geometry from two projections of algebraic curves.
2. Dimension analysis for the minimal number of algebraic curves required for a solution of the epipolar geometry.
3. Reconstruction from two views of a curve of degree  $d$  is a curve which contains two irreducible components one of degree  $d$  and the other of degree  $d(d-1)$  — a result that leads to a unique reconstruction of the original curve.
4. Formula for the minimal number of views required for the reconstruction of the dual curve.
5. Formula for the minimal number of views required for the reconstruction of the curve representation  $\mathbb{G}(1, 3)$ .

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