On calibration and reconstruction from planar curves

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Abstract

We describe in this paper closed-form solutions to the following problems in multi-view geometry of n'th order curves: (i) recovery of the fundamental matrix from 4 or more conic matches in two views, (ii) recovery of the homography matrix from a single n'th order ($n \geq 3$) matching curve and, in turn, recovery of the fundamental matrix from two matching n'th order planar curves, and (iii) 3D reconstruction of a planar algebraic curve from two views.

Although some of these problems, notably (i) and (iii), were introduced in the past [15, 3], our derivations are analytic with resulting closed form solutions. We have also conducted synthetic experiments on (i) and real image experiments on (ii) and (iii) with subpixel performance levels, thus demonstrating the practical use of our results.

1 Introduction

A large body of research has been devoted to the problem of computing the epipolar geometry from point correspondences. The theory of fundamental matrix and its robust numerical computation from point correspondences are well understood [16, 7, 12]. The next natural step has been to address the problem of lines or point-lines correspondences. It has been showed in that case three views are necessary to obtain constraints on the viewing geometry [28, 31, 32, 34, 13].

Since scenes rich with man-made objects contain curve-like features, the next natural step has been to consider higher-order curves. Given known projection matrices (or fundamental matrix and trifocal tensor) [23, 19, 20] show how to recover the 3D position of a conic section from two and three views, and [25] show how to recover the homography matrix of the conic plane, and [6, 30] shows how to recover a quadric surface from projections of its occluding conics. Reconstruction of higher-order curves were addressed in [3] and in [22, 8]. In [3] the matching curves are represented parametrically where the goal is to find a re-parameterization of each matching curve such that in the new parameterization the points traced on each curve are matching points. The optimization is over a discrete parameterization, thus, for a planar curve of degree n, which represented by $\frac{1}{2}n(n+3)$ points, one would need $n(n+3)$ minimal number of parameters to solve for in a non-linear bundle adjustment machinery — with some prior knowledge of a good initial guess. In [22,8] the reconstruction is done under infinitesimal motion assumption with the computation of spatio-temporal derivatives that minimize a set of non-linear equations at many different points along the curve. Finally, there have been attempts also [15] to recover the fundamental matrix from matching conics with the result that 4 matching conics are minimally necessary for a unique solution — albeit, the result is obtained by using a computer algebra system. The method developed there is specific to conics and is thus difficult to generalize to higher-order curves.

In this paper we treat the problems of recovering fundamental matrix, homography matrix, and 3D reconstruction (given fundamental matrix) using matching curves (represented in implicit form) of n'th order arising from planar n'th order curves. The emphasis in our approach is to produce closed form solutions. Specifically, we show the following three results:

1. We revisit the problem of recovering the fundamental matrix from matching conics [15] and re-prove, this time analytically, the result that 4 matching conics are necessary for a unique solution. We show that the equations necessary for proving this result are essentially the kruppa's equations [21] which are well known in the context of self calibration.
2. We show that the homography matrix of the plane of an algebraic curve of n'th order ($n \geq 3$) can be uniquely recovered from the projections of the curve, i.e., a single curve match between two images is sufficient for solving for the associated homography matrix. Our approach relies on inflection and singular points of the matching curves — the resulting procedure is simple and is closed-form.
3. We derive a simple algorithm(s) for reconstructing a planar algebraic curve of $n'th$ order from its projections. The algorithms are closed-form where the most "complicated" stage is finding the roots of a uni-variate polynomial.

We have conducted synthetic experiments on recovery of fundamental matrix from matching conics, and real imagery experiments on recovering the homography from a single matching curve of 3'rd order, and reconstruction of a 4'th order curves from two views. The later two experiments display subpixel performance levels, thus demonstrating the practical use of our results.

2 Background

Our algorithms are valid for planar algebraic curves. We start by presenting an elementary introduction to algebraic curves, and then some introductory properties about two images of the same planar curve useful for the rest of our work. More material can be found in [11].

2.1 Planar algebraic curves

We assume that the image plane is embedded into a projective plane. We assume that the ground field is the field of complex numbers. This makes the formulation simpler. But eventually we take into account only the real solutions.

Definition 1 Planar algebraic curve
A planar algebraic curve $\mathcal{C}$ is a subset of points, whose projective coordinates satisfy an homogeneous polynomial equation: $f(x, y, z) = 0$. The degree of f is called the order of $\mathcal{C}$. The curve is said to be irreducible, when the polynomial f cannot be divided by a non-constant polynomial.

We assume that all the curves we are dealing with are *planar irreducible algebraic curves*. Note that when two polynomials define the same irreducible curve, they must be equal up to scale. For convenience and shorter formulation, we define *a form $f \in \mathcal{C}[x, y, z]$ of degree n to be an homogeneous polynomial in x, y, z of total degree n.*

Let $\mathcal{C}$ be a curve of order n and let $\mathcal{L}$ be a given line. We can represent the line parametrically by taking two fixed points $\mathbf{a}$ and $\mathbf{b}$ on it, so that a general point $\mathbf{p}$ (except $\mathbf{b}$ itself) on it is given by $\mathbf{a} + \lambda\mathbf{b}$. The intersections of $\mathcal{L}$ and $\mathcal{C}$ are the points $\{\mathbf{p}_\lambda\}$, such that the parameters λ satisfy the equation:

$$J(\lambda) \equiv f(a_x + \lambda b_x, a_y + \lambda b_y, a_z + \lambda b_z) = 0$$

Taking the first-order term of the Using a Taylor-Lagrange expansion:

$$J(\lambda) = f(\mathbf{a}) + \lambda(\tfrac{\partial f}{\partial x}(\mathbf{a})b_x + \tfrac{\partial f}{\partial y}(\mathbf{a})b_y + \tfrac{\partial f}{\partial z}(\mathbf{a})b_z) = f(\mathbf{a}) + \lambda\nabla f(\mathbf{a}).\mathbf{b} = 0$$

If $f(\mathbf{a}) = 0$, $\mathbf{a}$ is located on the curve. Furthermore let assume that $\nabla f(\mathbf{a}).\mathbf{b} = 0$, then the line $\mathcal{L}$ and the curve $\mathcal{C}$ meet at $\mathbf{a}$ in two coincident points. A point is said to be *regular* is $\nabla f(\mathbf{a}) \neq \mathbf{0}$. Otherwise it is a *singular* (or *multiple*) point. When the point $\mathbf{a}$ is regular, *the line $\mathcal{L}$ is said to be tangent to the curve $\mathcal{C}$ at* $\mathbf{a}$.

Since the fundamental matrix is a mapping from the first image plane into the dual of the second image plane, which is the set of lines that lie on the second image, it will be useful to consider the following notion:

Definition 2 Dual curve
Given a planar algebraic curve $\mathcal{C}$, the dual curve is defined in the dual plane, as the set of all lines tangent to $\mathcal{C}$. The dual curve is algebraic and thus can be described as the set of lines (u, v, w), that are the zeros of a form $\phi(u, v, w) = 0$. If $\mathcal{C}$ is of order n, its dual curve $\mathcal{D}$ is of order less or equal to $n(n-1)$.

We will also need to consider the notion of inflexion point:

Definition 3 Inflexion point
An inflexion point $\mathbf{a}$ *of a curve $\mathcal{C}$ is a simple point of it whose tangent intersects the curve in at least three coincident points. This means that the third order term of the Taylor-Lagrange development must vanish too.*

It will be useful to compute the inflexion points. For this purpose we define the Hessian curve $\mathcal{H}(\mathcal{C})$ of $\mathcal{C}$, which is given by the determinantal equation:

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| = 0$$

It can be proven (see [27]) that the points where a curve $\mathcal{C}$ meets its Hessian curve $\mathcal{H}(\mathcal{C})$ are exactly the inflexion points and the singular points. Since the degree of $\mathcal{H}(\mathcal{C})$ is $3(n-2)$, there are $3n(n-2)$ inflexion and singular points counting with the corresponding intersection multiplicities (Bezout's theorem, see [27]).

2.2 Introductory properties

In this section, we are interested in providing a few general properties of two images of the same planar algebraic curve. First, note that the condition that the plane of the curve in space does not pass through the camera centers is equivalent to the fact that the curves in the image planes do not collapse to lines and are projectively isomorphic to the curve in space. Furthermore, the homography matrix induced by the plane of the curve in space is regular.

Proposition 1 Homography mapping
Let α be the mapping from the first image to the second image, that sends $\mathbf{p}$ to $\mathbf{Ap}$. Let $f(x,y,z)=0$ (respectively $g(x,y,z)=0$) be the equation of the curve $\mathcal{C}$ (respectively $\mathcal{C}'$) in the first (respectively second) image. We have the following constraint on the homography $\mathbf{A}$:

$$\exists\lambda,\ \forall x,y,z,\ g\circ\alpha(x,y,z)=\lambda f(x,y,z)$$

Proof: Since the curve $\mathcal{C}$ and $\mathcal{C}'$ are corresponding by the homography $\mathbf{A}$, the two irreducible polynomials $g\circ\alpha$ and f define the same curve $\mathcal{C}$. Thus these polynomials must be equal up to a scale factor (see previous subsection).

Proposition 2 Tangency conservation
Let $\mathcal{J}$ be the set of the epipolar lines in the first image that are tangent to the curve $\mathcal{C}$, and let be $\mathcal{J}'$ the set of epipolar lines in the second image that are tangent to the curve $\mathcal{C}'$. The elements of $\mathcal{J}$ and $\mathcal{J}'$ are in correspondence through the homography $\mathbf{A}$ induced by the plane of the curve in space.

Proof: Let f (respectively g) be the irreducible polynomial that defines $\mathcal{C}$ (respectively $\mathcal{C}'$). Let α be the mapping from the first image plane to the second image plane, that takes a point $\mathbf{p}$ and sends it to $\mathbf{Ap}$. According to the previous proposition, the two polynomials f and $g\circ\alpha$ are equal up to scale μ. Let $\mathbf{e}$ and $\mathbf{e}'$ be the two epipoles. Let $\mathbf{p}$ a point located on $\mathcal{C}$. The line joining $\mathbf{e}$ and $\mathbf{p}$, is tangent to $\mathcal{C}$ at $\mathbf{p}$ if $\lambda=0$ is a double root of the equation: $f(\mathbf{p}+\lambda\mathbf{e})=0$. (If $\mathbf{e}$ is located on $\mathcal{C}$, we invert $\mathbf{p}$ and $\mathbf{e}$.) This is equivalent to say that $\nabla f(\mathbf{p}).\mathbf{e}=0$. Since $\nabla g(\mathbf{p}').\mathbf{e}'=\nabla g(\mathbf{Ap}).\mathbf{Ae}=dg(\alpha(\mathbf{p}))\circ d\alpha(\mathbf{p}).\mathbf{e}=d(g\circ\alpha)(\mathbf{p}).\mathbf{e}=\mu df(\mathbf{p}).\mathbf{e}=\mu\nabla f(\mathbf{p}).\mathbf{e}=0$. Therefore it is equivalent to the tangency of the line $\mathbf{e}'\wedge\mathbf{Ap}$ with $\mathcal{C}'$. Given a line $\mathbf{l}\in\mathcal{J}$, its corresponding line $\mathbf{l}'\in\mathcal{J}'$ is given by: $\mathbf{A}^{-T}\mathbf{l}=\mathbf{l}'$. [1] □

[1] By duality $\mathbf{A}^T$ sends the lines of the second image plane into the lines of the first image plane. Here we have showed that $\mathbf{A}^T$ induces to one-to-one correspondence between $\mathcal{J}'$ and $\mathcal{J}$.

Note that since epipolar lines are transformed in the same way through any homography, the two sets $\mathcal{J}$ and $\mathcal{J}'$ are in fact projectively related by any homography. Some authors have already observed a similar property for apparent contours (see [1] and [2]).

Proposition 3 Inflexions and singularities conservation
The inflexions (respectively the singularities) of the two image curves are projectively related by the homography through the plane of the curve in space.

Proof: This double property is implied by the simple relations (we use the same notations than in the previous proposition):

$$\begin{bmatrix} \frac{\partial f}{\partial x}(\mathbf{p}) \\ \frac{\partial f}{\partial y}(\mathbf{p}) \\ \frac{\partial f}{\partial y}(\mathbf{p}) \end{bmatrix} = \mathbf{A}^T \begin{bmatrix} \frac{\partial g}{\partial x}(\mathbf{Ap}) \\ \frac{\partial g}{\partial y}(\mathbf{Ap}) \\ \frac{\partial g}{\partial y}(\mathbf{Ap}) \end{bmatrix} \quad .$$

$$[\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{p})] = \mathbf{A}^T [\frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{Ap})]\mathbf{A}$$

The first relations implies the conservation of the singularities by homography, whereas the second relation implies the conservation of the whole Hessian curve by homography.

3 Recovering the epipolar geometry from curve correspondences

3.1 From conic correspondences

Let $\mathbf{C}$ (respectively $\mathbf{C}'$) be the full rank (symmetric) matrix of the conic in the first (respectively second) image. The equations of the dual curves are $\phi(u, v, w) = \mathbf{l}^T\mathbf{C}^*\mathbf{l} = 0$ and $\psi(u, v, w) = \mathbf{l}^T\mathbf{C}'^*\mathbf{l} = 0$ where $\mathbf{l} = [u, v, w]^T$, $\mathbf{C}^\star = det(\mathbf{C})\mathbf{C}^{-1}$ and $\mathbf{C}'^\star = det(\mathbf{C}')\mathbf{C}'^{-1}$. $\mathbf{C}^\star$ and $\mathbf{C}'^\star$ are the adjoint matrices of $\mathbf{C}$ and $\mathbf{C}'$ (see [26]).

Theorem 1 *The fundamental matrix, the first epipole and the conic matrices are linked by the following relation:*

$$\exists\lambda \neq 0,\ \textit{such as: } \mathbf{F}^T\mathbf{C}'^\star\mathbf{F} = \lambda[\mathbf{e}]_x\mathbf{C}^\star[\mathbf{e}]_x, \tag{1}$$

where $[\mathbf{e}]_x$ *is the matrix that represents the linear map* $\mathbf{p} \longmapsto \mathbf{e} \wedge \mathbf{p}$.

Proof: According to proposition 2, both sides of the equation are in fact the two tangents of the conic $\mathcal{C}$, passing the epipole $\mathbf{e}$. Each tangent appears at the first order in both expression. Therefore they are equal up to a non-zero scale factor.

It is worthwhile noting that these equations are identical to Kruppa's equations [21] which were introduced in the context of self-calibration.

From equation 1, one can extract a set, denoted $\mathcal{E}_\lambda$, of six equations on $\mathbf{F}$, $\mathbf{e}$ and an auxiliary unknown λ. By eliminating λ it is possible to get five bi-homogeneous equations on $\mathbf{F}$ and $\mathbf{e}$.

Theorem 2 *The six equations, $\mathcal{E}_\lambda$, are algebraically independent.*

Proof: Using the following isomorphic mapping: $(\mathbf{F}, \mathbf{e}, \lambda) \longmapsto (\mathbf{D}'^{\star}\mathbf{F}\mathbf{D}^{-1}, \mathbf{D}\mathbf{e}, \lambda) = (\mathbf{X}, \mathbf{y}, \lambda)$, where $\mathbf{D} = \sqrt{C}$ and $\mathbf{D}'^{\star} = \sqrt{\mathbf{C}'^{\star}}$, in the field of complex, the original equations are mapped into the upper-triangle of $\mathbf{X}^T\mathbf{X} = \lambda[\mathbf{y}]_x^2$. Given this simplified form, it is possible of compute a Groebner basis ([5], [4]). Then we can compute the dimension of the affine variety in the variables $(\mathbf{X}, \mathbf{y}, \lambda)$, defined by these six equations. The dimension is 7, which shows that the equations are algebraically independent.

Note that the equations $\mathcal{E}_\lambda$ imply that $\mathbf{F}\mathbf{e} = \mathbf{0}$ (one can easily deduce it from the equation 1 [2]). In order to count the number of matching conics, in generic positions, that are necessary and sufficient to recover the epipolar geometry, we eliminate λ from $\mathcal{E}_\lambda$ and we get a set $\mathcal{E}$ that defines a variety $\mathcal{V}$ of dimension 7 in a 12-dimensional affine space, whose points are $(\mathbf{F}, \mathbf{e})$. The equations in $\mathcal{E}$ are bi-homogeneous in $\mathbf{F}$ and $\mathbf{e}$ and $\mathcal{V}$ can also be regarded as a variety of dimension 5 into the bi-projective space $\mathcal{P}^8 \times \mathcal{P}^2$, where $(\mathbf{F}, \mathbf{e})$ lie. Now we project $\mathcal{V}$ into $\mathcal{P}^8$, by eliminating $\mathbf{e}$ from the equation, we get a new variety $\mathcal{V}_f$ which is still of dimension 5 and which is contained into the variety defined by $det(\mathbf{F}) = 0$, whose dimension is 7 [3]. Therefore two pairs of matching conics in generic positions defines two varieties isomorphic to $\mathcal{V}_f$ which intersect in a three-dimensional variety $(5 + 5 - 7 = 3)$. A third conic in generic position will reduce the intersection to a one-dimensional variety $(5 + 3 - 7 = 1)$. A fourth conic will reduce the system to a zero-dimensional variety. These results can be compiled into the following theorem:

Theorem 3 *$\{$Four conics$\}$ or $\{$three conics and a point$\}$ or $\{$one conic and five points$\}$ in generic positions are sufficient to compute the epipolar geometry.*

We conclude this section by notifying that this dimensional result is valid under the assumption of complex varieties. Since we are interested in real solutions only, degeneracies might occur in very special cases such that then less than four conics might be sufficient to recover the epipolar geometry.

3.2 From higher order curve correspondences

Assume we have a projection of an n'th, $n \geq 3$, algebraic curve. We will show next that a single matching pair of curves are sufficient for uniquely recovering

[2] It is clear that we have: $\mathbf{F}^T\mathbf{C}'^{\star}\mathbf{F}\mathbf{e} = \mathbf{0}$. For any matrix $\mathbf{M}$, we have: $ker(\mathbf{M}^T) = Im(\mathbf{M})^T$. In addition, $\mathbf{C}'$ is invertible. Hence $\mathbf{F}\mathbf{e} = \mathbf{0}$

[3] Since it must be contained into the projection to $\mathcal{P}^8$ of the hypersurface defined by $det(\mathbf{F}\mathbf{e}) = 0$

the homography matrix induced by the plane of the curve in space, whereas two pairs of matching curves (residing on distinct planes) are sufficient for recovering the fundamental matrix.

Let $C_{i_1,\ldots,i_n}$ (respectively $C'_{i_1,\ldots,i_n}$) be the tensor form of the first (second) image curve. Let A^i_j be the tensor form of the homography matrix.

$$\exists \lambda \neq 0, \text{ such as: } C'_{i_1,\ldots,i_n} A^{i_1}_{j_1} \ldots A^{i_n}_{j_n} = \lambda C_{j_1,\ldots,j_n} \tag{2}$$

Since a planar algebraic curve of order n is represented by a polynomial containing $\frac{1}{2}n(n+3)+1$ terms, we are provided with $\frac{1}{2}n(n+3)$ equations (after elimination of λ) on the entries of the homography matrix. Let $\mathcal{S}$ denote this system. Therefore two curves of order $n \geq 3$ are in principle sufficient to recover the epipolar geometry. However we show a more geometric and more convenient way to extract the homography matrix since the system $\mathcal{S}$ might be very difficult to solve.

The simpler algorithm is true for non-oversingular curves, e.g. when a technical condition about the singularities of the curve holds. In order to make this condition explicit, we define a node to be an ordinary double point that is a double point with two distinct tangents, and a cusp to be a double point with coincident tangents. A curve of order n, whose only singular points are either nodes or cusps, satisfy the Plucker's formula (see [35]):

$$3n(n-2) = i + 6 \times \delta + 8 \times \kappa,$$

where i is the number of inflexion points, δ is the number of nodes, and κ is the number of cusps. For our purpose, a curve is said to be *non-oversingular* when its only singularities are nodes and cusps and when $i + s \geq 4$, where s is the number of all singular points.

Since the inflexion and singular points in both images are projectively related through the homography matrix (proposition 3), one can compute the homography through the plane of the curve in space of a curve of order $n \geq 3$, provided the previous condition holds. The resulting algorithm is as follows:

1. Compute the Hessian curves in both images.
2. Compute the intersection of the curve with its Hessian in both images. The output is the set of inflexion and singular points.
3. Discriminate between inflexion and singular points by the additional constraint for each singular point $\mathbf{a}$: $\nabla f(\mathbf{a}) = \mathbf{0}$.

At first sight, there are $i! \times s!$ possible correspondences between the sets of inflexion and singular points in the two images. But it is possible to further reduce the combinatorics by separating the points into two categories. The points are normalized such that the last coordinates is 1 or 0. Then separate real points from complex points. Each category of the first image must be matched with the same category in the second image. Then the right solution can be selected as it should be the one that makes the system $\mathcal{S}$ the closest to zero or the one that minimizes the Hausdorff distance (see [14]) between the set of points from the second image curve and the reprojection of the set of points from the first image curve into the second image. For better results, one can compute the Hausdorff distance on inflexion and singular points separately, within each category. We summarize this result:

Theorem 4 *The projections of a single planar algebraic curve of order $n \geq 3$ are sufficient for a unique solution for the homography matrix induced by the plane of the curve. The projections of two such curves, residing on distinct planes, are sufficient for a unique solution to the multi-view tensors (in particular to the fundamental matrix).*

It is worth noting that the reason why the fundamental matrix can be recovered from two pairs of curve matches is simply due to the fact that two homography matrices provide a sufficient set of linear equations for the fundamental matrix: if A_i, $i = 1, 2$, are two homography matrices induced by planes π_1, π_2, then $A_i^\top F + F^\top A_i = 0$ because $A_i^\top F$ is a symmetric matrix.

In the previous section, the computation of the epipolar geometry was intended using an equivalent to Kruppa's equations for any conic. It is of theoretical interest to investigate the question of possible generalization of Kruppa's equations to higher order curves. To this intent, let ϕ (respectively ψ) be the dual curve in the first (respectively second) image. Let γ (respectively ξ) be the mapping sending a point $\mathbf{p}$ from the first image into its epipolar $\mathbf{e} \wedge \mathbf{p}$ (respectively $\mathbf{Fp}$) in the first (respectively second) image. Then the theorem 1 holds in the general case, and can be regarded as an extended version of Kruppa's equation:

Theorem 5 *The dual curves in both images are linked by the following expression:*

$$\exists \lambda \neq 0, \ \textit{such as: } \psi \circ \xi = \lambda \phi \circ \gamma \tag{3}$$

Proof: According to their geometric interpretation, the sets defined by each side of this equation are identical. It is in fact the set of tangents to the first image curve, passing through the first epipole. It is left to show that each tangent appears with the same multiplicity in each representation. It is easily checked by a short computation, where $\mathbf{A}$ is the homography matrix between the two images, through the plane of the curve in space and $\alpha(\mathbf{p}) = \mathbf{Ap}$: $\psi \circ \xi(\mathbf{p}) = \psi(\mathbf{e}' \wedge \mathbf{Ap}) = \psi(\alpha(\mathbf{e}) \wedge \alpha(\mathbf{p})) \cong \psi \circ ({}^t\alpha)^{-1}(\mathbf{p})$. [4] Then it is sufficient to see that the dual formulation of the property 1 is written by $\psi \circ ({}^t\alpha)^{-1} \cong \phi$.

4 3D reconstruction

We turn our attention to the problem of reconstructing a planar algebraic curve from two views. Let the camera projection matrices be $[\mathbf{I}; \mathbf{0}]$ and $[\mathbf{H}; \mathbf{e}']$. We propose two simple algorithms.

[4] Indeed for a regular matrix $\mathbf{A}$: $\mathbf{Ax} \wedge \mathbf{Ay} = det(A)\mathbf{A}^{-T}(\mathbf{x} \wedge \mathbf{y})$. Then since ψ is a form, the last equality is true up to the scale factor, $det(A)^{deg(\psi)}$.

4.1 Algebraic approach

In this approach we first recover the homography matrix induced by the plane of the curve in space. This approach reduces the problem of finding the roots of uni-variate polynomials. The approach is inspired by the technique of recovering the homography matrix in [25]. It is known that any homography can be written as: $\mathbf{A} = \mathbf{H} + \mathbf{e}'\mathbf{a}^T$ (see [29, 17]). Using the equation 2, we get the following relation:

$$\mu^{n-1} C'_{i_1,\ldots,i_n} e'^{i_1} \ldots . e'^{i_{n-1}} A_j^{i_n} = \lambda C_{j_1,\ldots,j} e_1^i \ldots . e^{i_{n-1}},$$

where $\mathbf{Ae} = \mu \mathbf{e}'$.

Note that in fact this equation can be also obtained in a pure geometric way, by saying that the polar curve with respect to the epipole [11, 24, 27, 35] is conserved by homography. Let $\Delta_j = C_{j_1,\ldots,j} e_1^i \ldots . e^{i_{n-1}}$ and $\Delta'_{i_n} = C'_{i_1,\ldots,i_n} e'^{i_1} \ldots . e'^{i_{n-1}}$. Therefore we get: $\Delta'_{i_n}(H_j^{i_n} + e'^{i_n} a_j) = \frac{\lambda}{\mu^{n-1}} \Delta_j$. Thus the vector $\mathbf{a}$ can be expressed as a function of $\eta = \frac{\lambda}{\mu^{n-1}}$, (provided the epipoles are not located on the image curves): $a_j = \frac{1}{\beta}(\eta \Delta_j - \Delta'_k H_j^k)$, where $\beta = C'_{i_1,\ldots,i_n} e'^{i_1} \ldots . e'^{i_n}$. Then $A_j^i = H_j^i + \frac{1}{\beta}(\eta \Delta_j e'^i - \Delta'_k H_j^k e'^i)$. Substituting this expression of $\mathbf{A}$ into the equation 2 and eliminating λ leads to a set of equations of degree n on η. We are looking for the real common solutions.

In the conic case, there will in general two distinct real solutions for η corresponding to the two planar curves that might have produced the images. For higher order curve, the situation may be more complicated.

4.2 Geometric approach

This following approach highlights the geometric meaning of the reconstruction problem. The reconstruction is done in three steps:

1. Compute the cones generated by the camera centers and the image curves, whose equations are denoted F and G.
2. Compute the plane of the curve in space.
3. Compute the intersection of the plane with one of the cones.

The three steps are detailed below.

Computing the cones equations.

For a general camera, let $\mathbf{M}$ be the camera matrix. Let τ be the projection mapping from 3D to 2D: $\tau(\mathbf{P}) = \mathbf{MP}$. Let $f(\mathbf{p}) = 0$ be the equation of the image curve. Since a point of the cone is characterized by the fact that $f(\tau(\mathbf{P})) = 0$, the cone equation is simply: $f(\tau(\mathbf{P})) = 0$. Here we have; $F(\mathbf{P}) = f([\mathbf{I}; \mathbf{0}]\mathbf{P})$ and $G(\mathbf{P}) = f([\mathbf{H}; \mathbf{e}']\mathbf{P})$.

Computing the plane of the curve in space.

Theorem 6 *The plane equation $\pi(\mathbf{P}) = 0$ satisfies the following constraint. There exists a scalar k and a polynomial r, such that: $r \times \pi = F + kG$.*

Proof: F and G can be regarded as regular functions on the plane. Since they are irreducible polynomials and vanish on the plane on the same irreducible curve and nowhere else, they must be equal up to a scalar in the coordinate ring of the plane, e.g. they are equal up to a scalar modulo π.

Let $\pi(\mathbf{P}) = \alpha \times X + \beta \times Y + \gamma \times Z + \delta \times T$, where $\mathbf{P} = [X, Y, Y, T]^T$. The theorem provides $\binom{3+n}{n}$ (which is the number of terms in a polynomial that defines a surface of order n in the three-dimensional projective space) equations on $k, \alpha, \beta, \gamma, \delta, (r_i)_{1 \leq i \leq s}$, where $s = \binom{3+n-1}{n-1}$ and the $(r_i)_i$ are the coefficients of r. One can eliminate the auxiliary unknowns $k, (r_i)$, using Groebner basis [4], [5] or resultant systems [33], [18]. Therefore we get $1/2n(n+3)$ equations on $\alpha, \beta, \gamma, \delta$.

However, a more explicit way to perform this elimination follows. Let S be the surface, whose equation is $\Sigma = F + kG = 0$. The points $\mathbf{P}$ that lie on the plane π are characterized by the fact that when regarded as points of S, their tangent planes are exactly π. This is expressed by the following system of equations:

$$\left\{ \begin{array}{c} \pi(\mathbf{P}) = 0 \\ (\beta \frac{\partial \Sigma}{\partial X} - \alpha \frac{\partial \Sigma}{\partial Y})(\mathbf{P}) = 0 \\ (\gamma \frac{\partial \Sigma}{\partial X} - \alpha \frac{\partial \Sigma}{\partial Z})(\mathbf{P}) = 0 \\ (\delta \frac{\partial \Sigma}{\partial X} - \alpha \frac{\partial \Sigma}{\partial T})(\mathbf{P}) = 0 \end{array} \right. .$$

.

On the other hand, on the plane $\Sigma(\mathbf{P}) = F(\mathbf{P}) + kG(\mathbf{P}) = 0$. Therefore $k = -\frac{F(\mathbf{P})}{G(\mathbf{P})}$ for any $\mathbf{P}$ on the plane that is not located on the curve itself. Therefore we get the following system:

$$\left\{ \begin{array}{c} \pi(\mathbf{P}) = 0 \\ (\beta(G\frac{\partial F}{\partial X} - F\frac{\partial G}{\partial X}) - \alpha((G\frac{\partial F}{\partial Y} - F\frac{\partial G}{\partial Y}))(\mathbf{P}) = 0 \\ (\gamma(G\frac{\partial F}{\partial X} - F\frac{\partial G}{\partial X}) - \alpha((G\frac{\partial F}{\partial Z} - F\frac{\partial G}{\partial Z}))(\mathbf{P}) = 0 \\ (\delta(G\frac{\partial F}{\partial X} - F\frac{\partial G}{\partial X}) - \alpha((G\frac{\partial F}{\partial T} - F\frac{\partial G}{\partial T}))(\mathbf{P}) = 0 \end{array} \right.$$

Since the plane we are looking for doesn't pass through the point $[0, 0, 0, 1]^T$ which is the first camera center, δ can be normalized to 1. Thus for a point $\mathbf{P}$ on the plane, we have: $T = -(\alpha X + \beta Y + \gamma Z)$. By substituting this expression of T into the previous system, we get a new system that vanishes over all values of (X, Y, Z). Therefore its coefficients must be zero. This provides us with a large set of equations on (α, β, γ), that can be used to refine the solution obtained by the algebraic approach.

Computing the intersection of the plane and one of the cones.

The equation of the curve on the plane Π is given by the elimination of T between the two equations: $\alpha X + \beta Y + \gamma Z + T = 0$ and $f(\tau(\mathbf{P})) = 0$. Using the first cone gives us immediately the equation, since the first camera matrix is $[\mathbf{I};\mathbf{0}]$.

5 Experiments

5.1 Computing the epipolar geometry: 3 conics and 2 points

In order to demonstrate the validity of the theoretical analysis, we compute the fundamental matrix from 3 conics and 2 points in a synthetic experiment. The computation is too intense for the standard computer algebra packages. We have found that **Fast Gb** [5] a powerful program for Groebner basis, introduced by J.C. Faugere [9, 10] is one of the few packages that can handle this kind of computation. The conics in the first image are:

$f1(x,y,z) = x^2 + y^2 + 9z^2$

$g1(x,y,z) = 4\,x^2 + y^2 + 81\,z^2$

$h1(x,y,z) = (4\,x + y)\,x + (x - 1/2\,z)\,y + (-1/2\,y + z)\,z$

The conics in the second image are:

$f2(x,y,z) = -\frac{1}{100(-4+\sqrt{3})^2}(-1900x^2 + 800x^2\sqrt{3} - 1309y^2 + 400y^2\sqrt{3} + 9820yz\sqrt{3} - 16000yz - 72700z^2 + 40000z^2\sqrt{3})$

$g2(x,y,z) = \frac{1}{400(4489+400\sqrt{3})^2}(33036473600x^2 + 5732960000x^2\sqrt{3} + 332999600xy\sqrt{3} - 214463200xy - 73852000xz - 1384952000xz\sqrt{3} + 9091399981y^2 + 1771266080y^2\sqrt{3} - 16090386780yz\sqrt{3} + 10160177600yz + 556496242300z^2 + 141582592000z^2\sqrt{3})$

$h2(x,y,z) = -\frac{1}{400(-561+38\sqrt{3})^2}(-519504000x^2 + 48311700z^2 - 125749120xy\sqrt{3} + 43249920xy - 254646400xz\sqrt{3} - 6553140yz\sqrt{3} + 56456040yz + 68848000x^2\sqrt{3} + 1279651200xz - 272267400z^2\sqrt{3} + 2522418y^2\sqrt{3} - 298209y^2)$

Given just the constraints deduced from the conics, the system defines, as expected, a one-dimensional variety in $\mathcal{P}^8 \times \mathcal{P}^2$. When just one point is introduced, we get a zero-dimensional variety, whose degree is 516. When two points are introduced, the system reduces to the following:

$$\left\{ \begin{array}{c} F[1,1] = F[2,2] = F[2,3] = F[3,2] = F[3,3] = 0 \\ F[3,1] + (\sqrt{3}-1)F[1,3] = 0 \\ 10F[2,1] + (\sqrt{3}-1)F[1,3] = 0 \\ 10F[1,2] + (\sqrt{3}-2)F[1,3] = 0 \\ 133813 * F[1,3]^2 - 20600 * \sqrt{3} - 51100 = 0 \end{array} \right.$$

Then it is easy to get the right answer for the fundamental matrix:

$$\begin{bmatrix} 0 & -\frac{-2+\sqrt{3}}{\sqrt{511-206\sqrt{3}}} & 10\,\frac{1}{\sqrt{511-206\sqrt{3}}} \\ -\frac{-1+\sqrt{3}}{\sqrt{511-206\sqrt{3}}} & 0 & 0 \\ -10\,\frac{-1+\sqrt{3}}{\sqrt{511-206\sqrt{3}}} & 0 & 0 \end{bmatrix}$$

[5] https://fgb.medicis.polytechnique.fr/

5.2 Computing the homography matrix

We have performed a real image test on recovering the homography matrix induced by the plane of a 3'rd order curve. The equations of the image curves were recovered by least-squares fitting. Once the homography was recovered we used it to map the curve in one image onto its matching curve in the other image and measure the geometric distance (error). The error is at subpixel level which is a good sign to the practical value of our approach. Figure 1 displays the results.

Fig. 1. The first and the second image cubic.

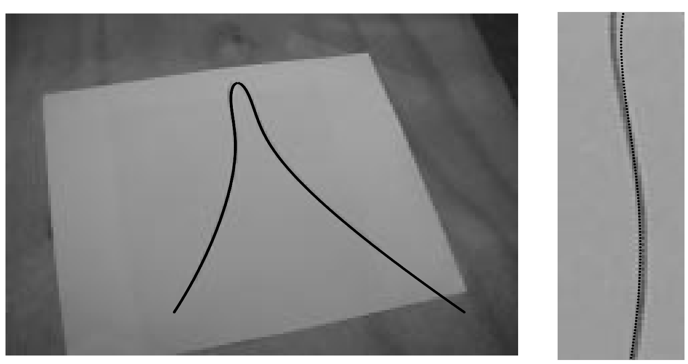

Fig. 2. The reprojected curve is overlayed on the second image cubic. A zoom shows the very slight difference.

5.3 3D reconstruction

Given two images of the same curve of order 4 (figure 3) and the epipolar geometry, we start by compute the plane and the homography matrix, using the algebraic approach to reconstruction. There are three solutions, that are all very robust. However to get further precision, one can refine it with the final system on the plane parameters, obtained at the end of the geometric approach. To demonstrate the accuracy of the algorithm, the reprojection of the curve in the second image is showed in the figure 4. The 3D rendering of the correct solution and the three solutions plotted together are showed in figure 5.

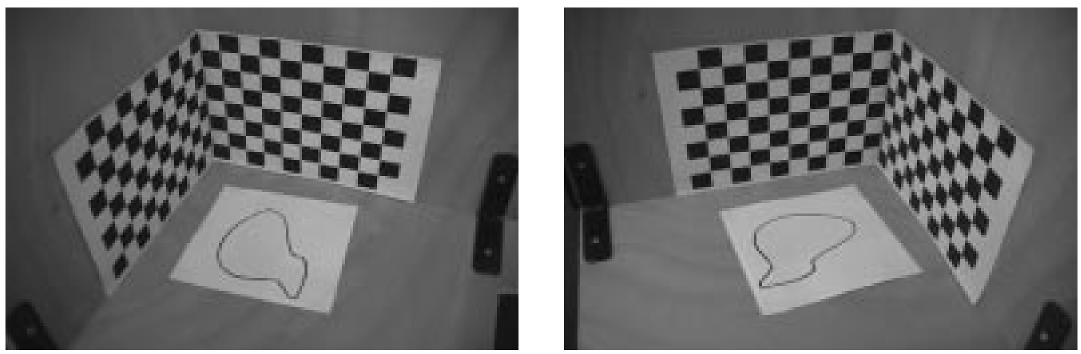

Fig. 3. The curves of order 4 as an input of the reconstruction algorithm.

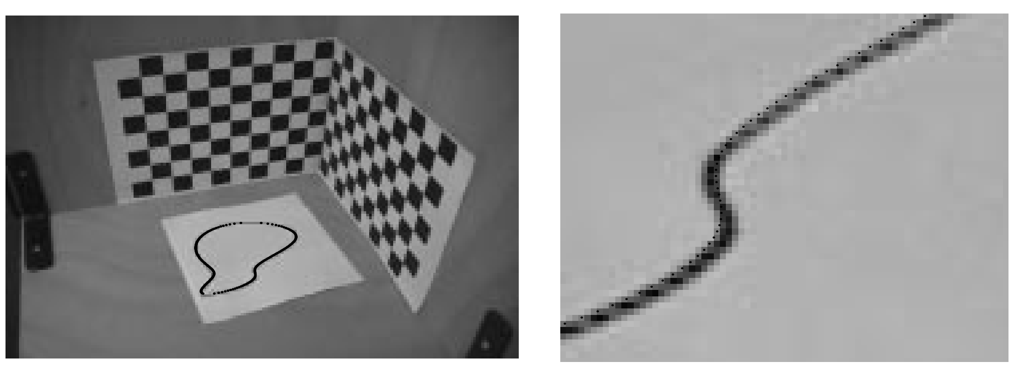

Fig. 4. Reprojection of the curve onto the second image.

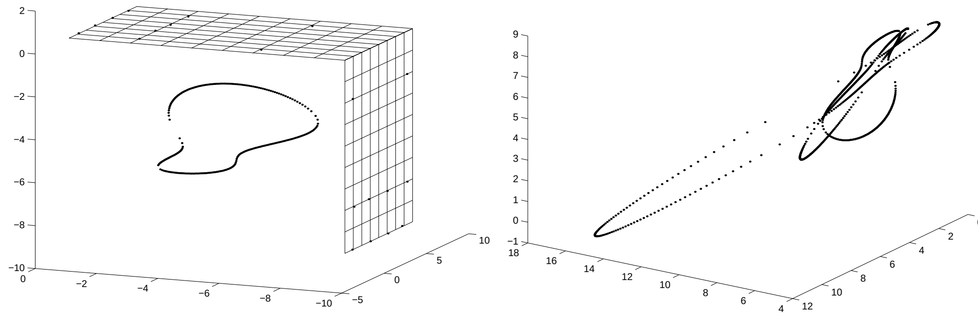

Fig. 5. The curves of order 4 as an input of the reconstruction algorithm.

Finally, the equation of the correct solution on its plane is given by: $f(x,y,z) =$
$$\frac{9006922504387547}{9007199254740992}\,z^4 - \frac{4947731105035649}{1152921504606846976}\,yz^3 + \frac{1070847909255857}{147573952589676412928}\,y^2z^2 - \frac{5458927196207623}{1208925819614629174706176}\,y^3z + \frac{3969428158337415}{2475880078570760549798248448}\,y^4 - \frac{7563069091264439}{1152921504606846976}\,xz^3 + \frac{5911661048544087}{295147905179352825856}\,xyz^2 - \frac{7447102119819593}{302231454903657293676544}\,xy^2z + \frac{3625625302714855}{618970019642690137449562112}\,xy^3 + \frac{4936178943362411}{295147905179352825856}\,x^2z^2 - \frac{8944822903795571}{302231454903657293676544}\,x^2yz + \frac{7158022235457567}{309485009821345068724781056}\,x^2y^2 - \frac{6146225343803339}{302231454903657293676544}\,zx^3 + \frac{7423176283805271}{618970019642690137449562112}\,x^3y + \frac{6539339092801811}{618970019642690137449562112}\,x^4$$

The curve is drawn on figure 6.

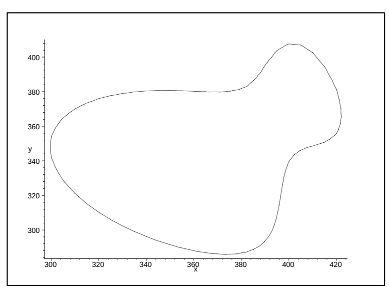

Fig. 6. The original curve.

6 Conclusion and future work

We have presented simple closed-form solutions for recovery of homography matrix from a single matching pair of curves of $n \geq 3$ order arising from a planar curve; two algorithms for reconstructing algebraic curves from their projections, again in closed-form; and revisited the problem of recovering the fundamental matrix from matching pairs of conics and proposed an analytic proof to the findings of [15] that four matching pairs are necessary for a unique solution.

Our experiments on real imagery demonstrate a sub-pixel performance level — an evidence to the practical value of our algorithms. Future work will investigate the same fundamental questions — calibration and reconstruction — from general three-dimensional curves.

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7 Appendix

7.1 Tensor representation of a planar algebraic curve

As a conic admits a matrix representation, $\mathbf{p}^T\mathbf{C}\mathbf{p} = 0$ iff $\mathbf{p}$ belongs to the conic, a general algebraic curve of order n admits a tensor representation: $\mathbf{T}_{i_1...i_n}\mathbf{p}^{i_1}...\mathbf{p}^{i_n} = 0$ iff $\mathbf{p}$ belongs to the curve, where for each k, $i_k \in 1, 2, 3$. In this tensor representation, a short notation is used: a repeated index on low and high position is summed over its domain definition. One has to link this tensor representation with the regular polynomial representation: $f(x, y, z) = 0$.

Lemma 1. *Let $f(x,y,z) = 0$ be an homogeneous equation of order n. There exists a tensor of order n, defined up to a scale factor, such as the equation can be rewritten in the following form:*

$$\mathbf{T}_{i_1 \dots i_n} \mathbf{p}^{i_1} \dots \mathbf{p}^{i_n} = 0,$$

where for each k, $i_k \in 1, 2, 3$, $\mathbf{p} = [x, y, z]^T$, and:

$$\mathbf{T}_{i_1 \dots i_n} = \mathbf{T}_{i_{\tau(1)} \dots i_{\tau(n)}}, \text{ for each } \tau \text{ which is a transposition of } \{1, 2, .., n\}.$$ [6]

Proof: The proof is quite forward. It is just necessary to remark that for each n-uplet $i_1, .., i_n$ such as: $i_1 \geq i_2 \geq ... \geq i_n$, we have: $T_{i_1 \dots i_n} = \alpha \frac{1}{a!b!c!} \frac{\partial^{a+b+c}}{\partial x^a y^b z^c} f(x, y, z)$, where: $a = \sum_{i_k=1} 1$, $b = \sum_{i_k=2} 1$, $c = \sum_{i_k=3} 1$ and $\alpha = \frac{a!b!c!}{n!}$. The factor α is due to the symmetry of the tensor. Finally $T_{i_1 \dots i_n} = \frac{1}{n!} \frac{\partial^{a+b+c}}{\partial x^a y^b z^c} f(x, y, z)$.

[6] A transposition of $\{1, 2, .., n\}$ is defined by the choice of a pair $\{i, j\} \subset \{1, 2, ..n\}$, such as: $i \neq j$ and $\tau(k) = k$ for each $k \in \{1, 2, ..n\} \setminus \{i, j\}$ and $\tau(i) = j$ and $\tau(j) = i$.