Multi-linear Systems and Invariant Theory

in the Context of Computer Vision and Graphics

Class 5: Self Calibration

CS329
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Material We Will Cover Today

- The basic equations and counting arguments
- The “absolute conic” and its image.
- Kruppa’s equations
- Recovering internal parameters.

The Basic Equations and Counting Arguments

Recall,

$$p \cong K[R;T]\begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

3D->2D from Euclidean world frame to image

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R\begin{pmatrix} U \\ V \\ W \end{pmatrix} + T$$

world frame to first camera frame

Let K,K' be the internal parameters of camera 1,2 and choose canonical frame in which R=I and T=0 for first camera.

$$p \cong [K; \quad 0]P$$

$$p' \cong K'[R \quad t]P$$

$$P = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

The Basic Equations and Counting Arguments

$$p \cong [I; \quad 0]WW^{-1}\begin{pmatrix} p \\ \mu \end{pmatrix}$$

$$p = \frac{1}{Z}K\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where $\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \cong W^{-1}\begin{pmatrix} p \\ \mu \end{pmatrix}$

$$W = \begin{bmatrix} K & 0 \\ n^T & 1 \end{bmatrix}$$

(8 unknown parameters)

W^{-1} maps from the projective frame to Euclidean

The Basic Equations and Counting Arguments

$(X, Y, Z, 0)$ are the points on the plane at infinity (in Euc frame)

$\Rightarrow$ $(0,0,0,1)$ is the plane at infinity

$$\pi_\infty = W^{-T} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ is the plane at infinity in Proj frame

(recall: if W maps points to points (Euc -> Proj), then the dual W^{-T} maps planes to planes)

The Basic Equations and Counting Arguments

$$\pi_\infty = W^{-T}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{bmatrix}K^{-T} & -K^{-T}n\\ 0^T & 1\end{bmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}-K^{-T}n\\1\end{pmatrix} = \begin{pmatrix}u\\1\end{pmatrix}$$

$$\Rightarrow \quad n = -K^T u$$

The Basic Equations and Counting Arguments

$$p' \cong \begin{bmatrix} H & e' \end{bmatrix} \begin{pmatrix} p \\ \mu \end{pmatrix}$$ Projective frame

$$= \begin{bmatrix} H & e' \end{bmatrix} W W^{-1} \begin{pmatrix} p \\ \mu \end{pmatrix}$$

$$= \begin{bmatrix} H & e' \end{bmatrix} \begin{bmatrix} K & 0 \\ n^T & 1 \end{bmatrix} W^{-1} \begin{pmatrix} p \\ \mu \end{pmatrix}$$

$$= K' \begin{bmatrix} R & t \end{bmatrix} W^{-1} \begin{pmatrix} p \\ \mu \end{pmatrix}$$

$$\Rightarrow \quad K' \begin{bmatrix} R & t \end{bmatrix} \cong \begin{bmatrix} H & e' \end{bmatrix} \begin{bmatrix} K & 0 \\ n^T & 1 \end{bmatrix}$$

The Basic Equations and Counting Arguments

$$K'\begin{bmatrix} R & t \end{bmatrix} \cong \begin{bmatrix} H & e' \end{bmatrix} \begin{bmatrix} K & 0 \\ n^T & 1 \end{bmatrix}$$

$$\Rightarrow \quad HK + e'n^T \cong K'R$$

$$K'^{-1}HK + K'^{-1}e'n^T \cong R$$

since $n = -K^T u$ then,

$$K'^{-1}HK - K'^{-1}e'u^T K \cong R$$

but $RR^T \cong I$ provides 5 (non-linear) constraints!

The Basic Equations and Counting Arguments

$$K'^{-1} HK - K'^{-1} e'u^T K \cong R$$

$$RR^T \cong I \quad \Rightarrow$$

$$\boxed{(H - e'u^T)KK^T(H - e'u^T)^T \cong K'K'^T}$$

Since the right-hand side is symmetric and up to scale, we have 5 constraints.

The Basic Equations and Counting Arguments

Lets do some counting:

Let $1 \leq \# K \leq 5$ be the number of internal parameters

m be the number of views

$$5(m-1) \geq (\# K) + (m-1)(\# K) + 3$$

The Basic Equations and Counting Arguments

$$5(m-1) \geq (\# K) + (m-1)(\# K) + 3$$

$\# K = 5 \Rightarrow$ not enough measurements (!)

$\# K = 4 \Rightarrow 5(m-1) \geq 7 + 4(m-1) \Rightarrow m \geq 8$

$\# K = 3 \Rightarrow 5(m-1) \geq 6 + 3(m-1) \Rightarrow m \geq 4$

$\# K = 1 \Rightarrow 5(m-1) \geq 4 + (m-1) \Rightarrow m \geq 2$

$K = K' = K''.... \Rightarrow 5(m-1) \geq 8 \Rightarrow m \geq 3$

(fixed internal params)

The remainder of this lecture is about a geometric insight of

$$\boxed{(H - e'u^T)KK^T(H - e'u^T)^T \cong K'K'^T}$$

The Absolute Conic

$p^T C p = 0$ where $C = C^T$ represents a conic in 2D

$(x_1, x_2, x_3, 0)$ are the points on the plane at infinity (in Euc frame)

$\Rightarrow$ $(0,0,0,1)$ is the plane at infinity

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T C \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$ is conic on the plane at infinity

when $C = I$ $\Rightarrow$ $x_1^2 + x_2^2 + x_3^2 = 0$

is the "absolute" conic (imaginary circle) Ω_∞

The Absolute Conic

Plane at infinity is preserved under affine transformations:

$$W = \begin{bmatrix} A & v \\ 0^T & 1 \end{bmatrix} \quad \text{because} \quad W \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ 0 \end{pmatrix}$$

Ω_∞ is preserved under similarity transformation (R,t up to scale)

$$W \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

if $p^T C p = 0$ and $Ap \cong p'$

then $p'^T C' p' = p^T (A^T C' A) p = 0$

$$\Rightarrow \quad C' = A^{-T} C A^{-1}$$

but $C = I$, so in order that $C' = I$ we must have:

$$I \cong A^{-T} A^{-1} \quad \Rightarrow \quad A \text{ is orthogonal}$$

The Image of the Absolute Conic

Image of points at infinity:

$$K[R \quad t]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = KR\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

let $A = KR$

if $p^T C p = 0$ is a conic on the plane at infinity

then $C' = A^{-T} C A^{-1}$ is the projected conic onto the image

since $C = I$, then $C' = (KR)^{-T}(KR)^{-1} = K^{-T}K^{-1}$

$\Rightarrow$ the image of Ω_∞ is

$$\boxed{\omega = K^{-T}K^{-1}}$$

The Image of the Dual Absolute Conic

$$p^T C p = 0 \quad \Rightarrow \quad l = Cp$$ is tangent to the conic at p

$$p = C^{-1} l \quad \Rightarrow \quad p^T C p = l^T C^{-T} C C^{-1} l = l^T C^{-1} l = 0$$

$$\Rightarrow \quad \boxed{\omega^* = K K^T}$$ is the image of the dual absolute conic

The basic equation:

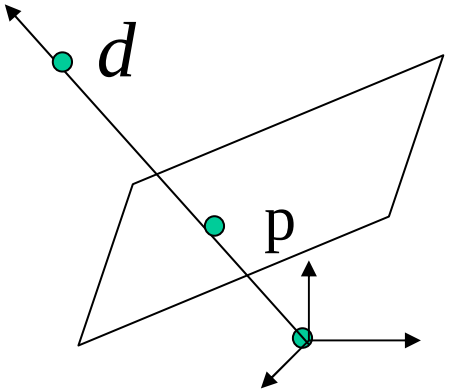
$$(H - e' u^T) K K^T (H - e' u^T)^T \cong K' K'^T$$

Becomes:

$$(H - e' u^T) \omega^* (H - e' u^T)^T \cong \omega'^*$$

Why 8 parameters? 5 for the conic, 3 for the plane

Geometric Interpretation of ω



$$\begin{pmatrix} d \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$p \cong \begin{bmatrix} K & 0 \end{bmatrix} \begin{pmatrix} \lambda d \\ 1 \end{pmatrix} \cong Kd \quad \Rightarrow \quad d = K^{-1} p$$

direction of optical ray

The angle between two optical rays d_1, d_2

$$\cos(\theta) = \frac{d_1^T d_2}{|d_1||d_2|} = \frac{p_1^T \omega p_2}{\sqrt{p_1^T \omega p_1}\sqrt{p_2^T \omega p_2}}$$

$\Rightarrow$ given $\omega = K^{-T} K^{-1}$ one can measure angles

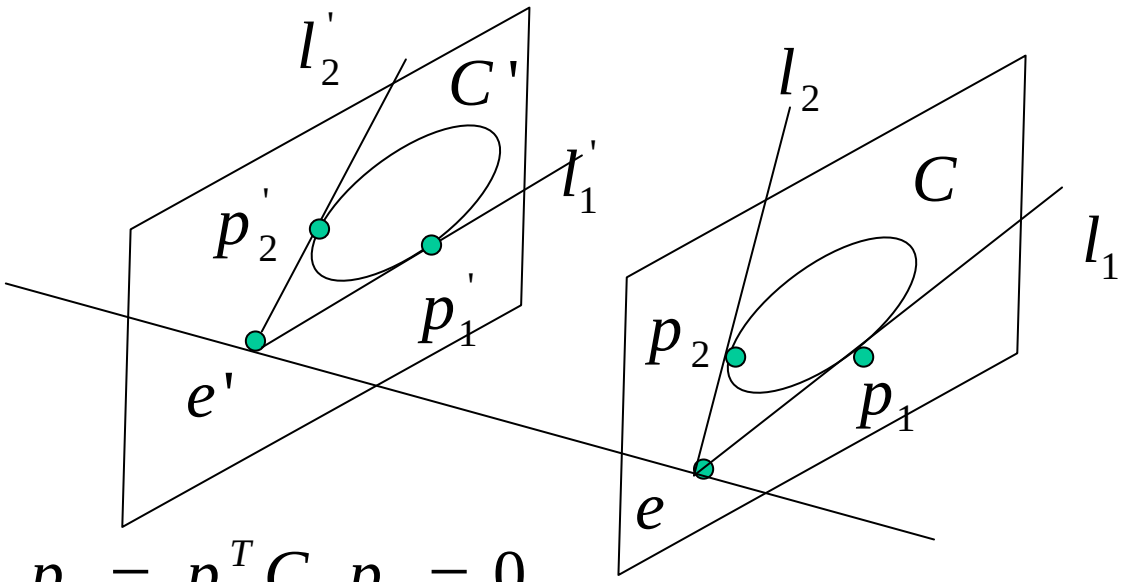
Kruppa's Equations

General idea: eliminate n from the basic equation.

$$p_i^T C p_i = 0$$

$$l_i^T C^{-1} l_i = 0$$

$$l_i = [e]_\times p_i$$



$$p_i^T [e]_\times C^{-1} [e]_\times p_i = p_i^T C_t p_i = 0$$

$$p_i'^T [e']_\times C'^{-1} [e']_\times p_i' = p_i'^T C_t' p_i' = 0$$

C_t, C_t' are degenerate (rank 2) conics

Kruppa's Equations

Note:

$$C_t = l_1 l_2^T + l_2 l_1^T \qquad \text{is a degenerate conic}$$

$$p^T C_t p = 0 \qquad \text{iff} \qquad p^T l_1 = 0 \qquad \text{or} \qquad p^T l_2 = 0$$

Let H_π be the homography induced by the plane of the conic

$$C_t \cong H^T C_t' H \qquad \text{(slide 14)}$$

$$\Rightarrow \qquad \underbrace{[e]_\times C^{-1} [e]_\times}_{C_t} \cong H_\pi^T \underbrace{[e']_\times C'^{-1} [e']_\times}_{C_t'} H_\pi$$

Kruppa's Equations

$$[e]_\times C^{-1}[e]_\times \cong H_\pi^T [e']_\times C'^{-1}[e']_\times H_\pi$$

Recall: $F = [e']_\times H_\pi$

$$\Rightarrow \quad [e]_\times C^{-1}[e]_\times \cong F^T C'^{-1} F$$

In our case $\pi = \infty$ and the conic is Ω_∞

$$\Rightarrow \quad \boxed{[e]_\times \omega^* [e]_\times \cong F^T \omega'^* F}$$

$$\boxed{[e]_\times KK^T [e]_\times \cong F^T K' K'^T F}$$

Likewise: $\boxed{[e']_\times \omega'^* [e']_\times \cong F \omega^* F^T}$

Determining K given π_∞

Recall: $\pi_\infty = \begin{pmatrix} u \\ 1 \end{pmatrix}$ the location of the plane at inifinity in the projective coordinate frame.

We wish to represent the homography H_∞ induced by π_∞

Let $P = \begin{pmatrix} p \\ \mu \end{pmatrix}$ $P^T \pi_\infty = 0$ be a point on the plane at infinity.

$$\Rightarrow \quad p^T u + \mu = 0 \quad \Rightarrow \quad P = \begin{pmatrix} p \\ -p^T u \end{pmatrix}$$

$$p' \cong [H \quad e'] \begin{pmatrix} p \\ \mu \end{pmatrix} = Hp - (p^T u)e' = (H - e'u^T)p$$

$$\Rightarrow \quad \boxed{H_\infty \cong H - e'u^T}$$

Determining K given π_∞

Recall: $$(H - e'u^T)\omega^*(H - e'u^T)^T \cong \omega'^*$$ (slide 16)

$$\omega^* = KK^T \qquad \omega'^* = K'K'^T$$

$$\Rightarrow \quad \boxed{H_\infty \omega^* H_\infty^T \cong \omega'^*}$$

Note: this could be derived from "first principles" as well:

$$l^T \omega^* l = 0$$
$$l'^T \omega'^* l' = 0$$

tangents lines to the image of the absolute conic

$$H_\infty^{-T} l \cong l' \quad \Rightarrow \quad l^T \underbrace{H_\infty^{-1} \omega'^* H_\infty^{-T}}_{\omega^*} l = 0$$

Determining K given π_∞

$$H_\infty \omega^* H_\infty^T \cong \omega'^*$$

Assume fixed internal parameters $\omega^* = \omega'^*$

$$H_\infty \omega^* H_\infty^T = \omega^* \qquad \det(H_\infty) = 1$$

Note: $\det(H_\infty) = 1 \Rightarrow H_\infty = KRK^{-1}$

$$KRK^{-1} KK^T K^{-T} R^T K^T = KK^T$$

Provides 4 independent linear constraints on $\omega^* = KK^T$

Why 4 and not 5?

$\Rightarrow$ we need 3 views (since ω^* has 5 unknowns)

Why 4 Constraints?

$$H_\infty \omega^* H_\infty^T = \omega^*$$

$$\det(H_\infty) = 1 \Longrightarrow H_\infty = KRK^{-1}$$

H_∞ and R are "similar" matrices, i.e., have the same eigenvalues

Let w be the axis of rotation, i.e., $Rw = w$

$\Longrightarrow$ H_∞ has an eigenvalue = 1, with eigenvector $v = Kw$

$$(KRK^{-1}v = v \Longrightarrow R(K^{-1}v) = K^{-1}v)$$

Why 4 Constraints?

$$H_\infty v = v$$

if ω^* is a solution to $H_\infty \omega^* H_\infty^T = \omega^*$

then $\omega^* + \lambda v v^T$ is also a solution

$$H_\infty(\omega^* + \lambda v v^T) H_\infty^T = H_\infty \omega^* H_\infty^T + \lambda (H_\infty v)(H_\infty v)^T$$

$$= \omega^* + \lambda v v^T$$

→ We need one more camera motion (with a different axis of rotation).

Kruppa's Equations (revisited)

Kruppa's equations: $[e']_\times \omega'^* [e']_\times \cong F \omega^* F^T$

Start with the basic equation:

$$H_\infty \omega^* H_\infty^T \cong \omega'^*$$

Multiply the terms by $[e']_\times$ on both sides:

$$[e']_\times \omega'^* [e']_\times \cong \underbrace{[e']_\times H_\infty}_{F} \omega^* H_\infty^T [e']_\times$$