Multi-linear Systems and Invariant Theory

in the Context of Computer Vision and Graphics

Class 3: Infinitesimal Motion

CS329
Stanford University

Amnon Shashua

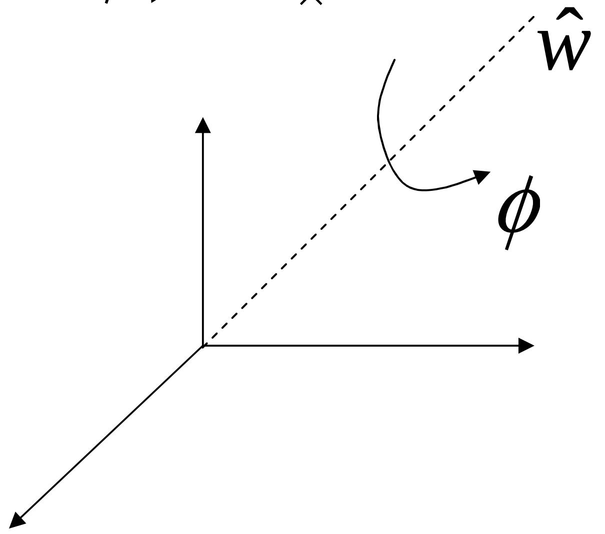
Material We Will Cover Today

- Infinitesimal Motion Model
- Infinitesimal Planar Homography (8-parameter flow)
- Factorization Principle for Motion/Structure Recovery
- Direct Estimation

Infinitesimal Motion Model

Rodriguez Formula:

$$R = (\cos\phi) I + (1 - \cos\phi)\hat{w}\hat{w}^T + (\sin\phi)[\hat{w}]_\times$$



$$\phi = d\phi \rightarrow 0$$

$$\cos(d\phi) \rightarrow 1$$

$$\sin(d\phi) \rightarrow 0$$

$$\Rightarrow \quad R \rightarrow I + (d\phi)[\hat{w}]_\times = I + [w]_\times$$

Infinitesimal Motion Model

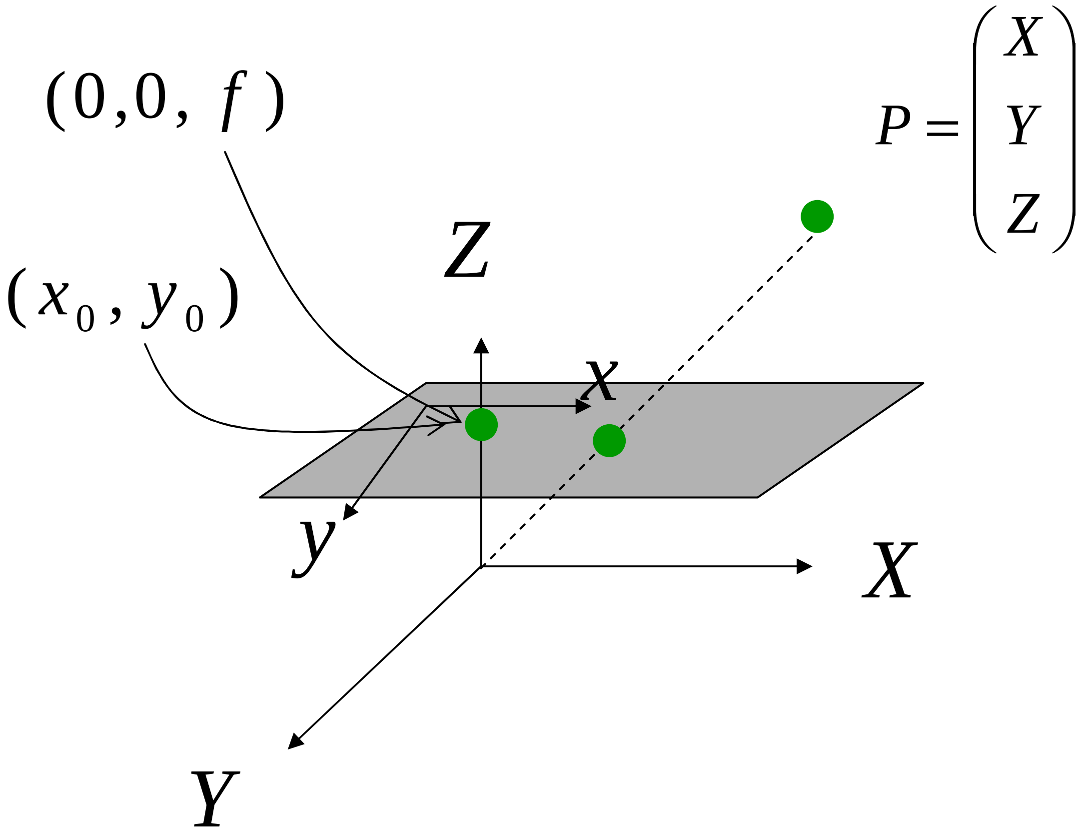
$$P' = RP + t$$
$$= (I + [w]_x)P + t$$

$$P = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\dot{P} = \frac{dP}{dt} \approx P' - P$$

$\Longrightarrow$ $$\boxed{\dot{P} = [w]_x P + t}$$

Reminder:



$$x = f\frac{X}{Z} + x_0$$

$$y = f\frac{Y}{Z} + y_0$$

Assume: $f = 1, x_0 = y_0 = 0$

$$\Rightarrow \quad p = \frac{1}{Z}P$$

Infinitesimal Motion Model

$$\dot{P} = [w]_x P + t$$

$$u = \frac{dx}{dt} = \frac{d}{dt}\left(\frac{X}{Z}\right) = \frac{\dot{X}Z - \dot{Z}X}{Z^2} = \frac{1}{Z}\left(\dot{X} - x\dot{Z}\right)$$

Let $s = (1, 0, -x)^T$ $\Rightarrow$ $u = \frac{1}{Z} s^T \dot{P}$

$\Rightarrow$

$$u = \frac{1}{Z} s^T t + s^T [w]_x p$$

Infinitesimal Motion Model

$$u = \frac{dx}{dt} = \frac{1}{Z} s^T t + s^T [w]_x p$$

$$s = (1,0,-x)^T$$

$$r = (0,1,-y)^T$$

$$p = (x, y, 1)^T$$

$$v = \frac{dy}{dt} = \frac{1}{Z} r^T t + r^T [w]_x p$$

Infinitesimal Planar Motion (the 8-parameter flow)

$$n^T P = 1 \quad \Longrightarrow \quad aX + bY + cZ = 1$$

$$\Longrightarrow \quad ax + by + c = \frac{1}{Z} \qquad Z \neq 0$$

$$u = (ax + by + c)s^T t + s^T [w]_x p$$

$$v = (ax + by + c)r^T t + r^T [w]_x p$$

Infinitesimal Planar Motion (the 8-parameter flow)

$$u = (ax + by + c)s^T t + s^T [w]_x p$$

$$= (ax + by + c)(t_1 - xt_3) + (1,0,-x)[w]_x p$$

$$u = (w_2 + ct_1) + (at_1 - ct_3)x + (bt_1 - w_3)y$$

$$+ (-bt_3 - w_1)xy + (w_2 - at_3)x^2$$

$$v = (-w_1 + ct_2) + (at_2 + w_3)x + (bt_2 - ct_3)y$$

$$+ (-bt_3 - w_1)y^2 + (w_2 - at_3)xy$$

Infinitesimal Planar Motion
(the 8-parameter flow)

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_7 xy + \alpha_8 x^2$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y + \alpha_8 xy + \alpha_7 y^2$$

Note: unlike the discrete case, there is no scale factor

Irani ICCV99
Shashua-Levin ECCV'00

Reconstruction of Structure/Motion (factorization principle)

$$u = \frac{1}{Z} s^T t + s^T [w]_x p$$

$$v = \frac{1}{Z} r^T t + r^T [w]_x p$$

$$s^T [w]_x p = s^T (w \times p) = w^T (p \times s)$$

Note:

$$a^T (b \times c) = [abc] = [bca] = [cab]$$ 2 interchanges

$$= -[bac] = -[cba] = -[acb]$$ 1 interchanges

Reconstruction of Structure/Motion (factorization principle)

$$u = \frac{1}{Z} s^T t + s^T [w]_x \, p$$

$$s^T [w]_x \, p = w^T (p \times s)$$

$$v = \frac{1}{Z} r^T t + r^T [w]_x \, p$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} p \times s & \frac{1}{Z} s^T \\ p \times r & \frac{1}{Z} r^T \end{bmatrix}_{2 \times 6} \begin{pmatrix} w \\ t \end{pmatrix}$$

Reconstruction of Structure/Motion (factorization principle)

Let (u_{ij}, v_{ij}) be the "flow" of point i at image j (image 0 is ref frame)

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ . & . & . & . \\ . & . & . & . \\ u_{n1} & u_{n2} & \cdots & u_{nm} \\ v_{11} & v_{12} & \cdots & v_{1m} \\ . & . & . & . \\ . & . & . & . \\ v_{n1} & v_{n2} & \cdots & v_{nm} \end{bmatrix} = \begin{bmatrix} p_1 \times s_1 & (1/Z)s_1^T \\ . & . \\ . & . \\ p_n \times s_n & (1/Z)s_n^T \\ p_1 \times r_1 & (1/Z)r_1^T \\ . & . \\ . & . \\ p_n \times r_n & (1/Z)r_n^T \end{bmatrix}_{2n\times 6} \begin{bmatrix} w_1 & \cdots & w_m \\ t_1 & \cdots & t_m \end{bmatrix}_{6\times m}$$

$$W = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} S_x \\ S_y \end{bmatrix} M = SM$$

Reconstruction of Structure/Motion (factorization principle)

$$W = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} S_x \\ S_y \end{bmatrix} M = SM$$

Given W, find S,M

Let $W = KL$ (using SVD)

$= (KA)(A^{-1}L) = SM$ for some $A_{6\times 6}$

Goal: find $A_{6\times 6}$ such that $KA = S$

using the "structural" constraints on S

Reconstruction of Structure/Motion (factorization principle)

Goal: find $A_{6\times 6}$ such that $KA = S$

using the "structural" constraints on S

Columns 1-3 of S are known, thus columns 1-3 of A can be determined.

Columns 4-6 of A contain 18 unknowns:

$(1/Z)s^T, (1/Z)r^T$ eliminate Z and one obtains 5 constraints

Reconstruction of Structure/Motion (factorization principle)

Goal: find $A_{6\times 6}$ such that $KA = S$

using the "structural" constraints on S

Let $K = \begin{bmatrix} K_x \\ K_y \end{bmatrix}_{2n\times 6}$ $A = [A_1, \ldots, A_6]$

$K_x A_5 = 0, K_y A_4 = 0$ because

$$(1/Z)(1,0,-x), (1/z)(0,1,-y)$$

Reconstruction of Structure/Motion (factorization principle)

$$K_x A_5 = 0, K_y A_4 = 0$$

because

$$(1/Z)(1,0,-x), (1/z)(0,1,-y)$$

$$K_x A_4 = K_y A_5$$

$$\frac{(K_x A_6)_i}{(K_x A_4)_i} = -x_i$$

→ Each point provides 5 constraints, thus we need 4 points and 7 views

$$\frac{(K_y A_6)_i}{(K_y A_5)_i} = -y_i$$

Direct Estimation

$$I_1 \equiv f(x, y)$$
$$I_2 \equiv g(x, y)$$

The grey values of images 1,2

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \begin{pmatrix} x + u \\ y + v \end{pmatrix}$$

Goal: find u,v per pixel

$$S(\hat{u}, \hat{v}) = \sum_{(x,y)\in R} [g(x - \hat{u}, y - \hat{v}) - f(x, y)]^2$$

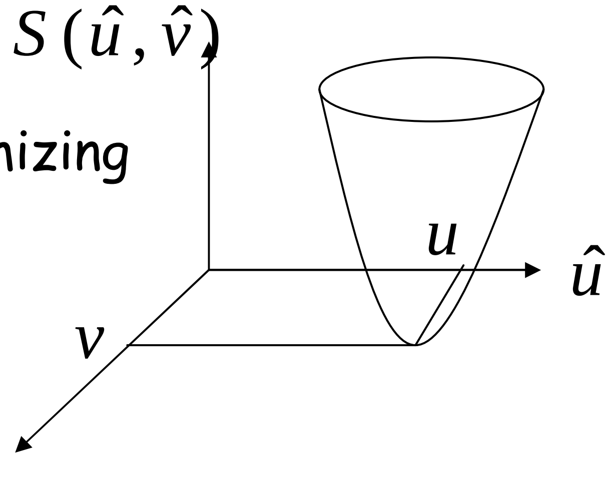
Direct Estimation

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \begin{pmatrix} x+u \\ y+v \end{pmatrix}$$

$$S(\hat{u},\hat{v}) = \sum_{(x,y)\in R} [g(x-\hat{u}, y-\hat{v}) - f(x,y)]^2$$

Assume: $$(u,v) = \arg\min_{\hat{u},\hat{v}} S(\hat{u},\hat{v})$$

We are assuming that (u,v) can be found by correlation principle (minimizing the sum of square differences).



Direct Estimation

$$S(\hat{u},\hat{v}) = \sum_{(x,y)\in R} [g(x-\hat{u}, y-\hat{v}) - f(x,y)]^2$$

Taylor expansion:

$$g(x-u, y-v) = g(x,y) - ug_x(x,y) - vg_y(x,y) + O(\Delta^2)$$

$$S(\hat{u},\hat{v}) \approx \sum_{(x,y)\in R} [\hat{u}g_x + \hat{v}g_y + I_t]^2$$

$$I_t(x,y) = f(x,y) - g(x,y)$$

Direct Estimation

$$S(\hat{u},\hat{v}) \approx \sum_{(x,y)\in R} [\hat{u}I_x + \hat{v}I_y + I_t]^2$$

I_x, I_y gradient of image 2

I_t image 1 minus image 2

$$\min_{u,v} \sum_{(x,y)\in R} [uI_x + vI_y + I_t]^2$$

$$A = \begin{bmatrix} I_x & I_y \\ . & . \\ . & . \\ . & . \\ I_x & I_y \end{bmatrix} \qquad x = \begin{pmatrix} u \\ v \end{pmatrix} \qquad b = \begin{pmatrix} I_t \\ . \\ . \\ . \\ I_t \end{pmatrix}$$

Direct Estimation

$$\min_{u,v} \sum_{(x,y)\in R} [uI_x + vI_y + I_t]^2 \iff \min_x |Ax - b|^2$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{pmatrix}$$

"aperture problem" $rank(A^T A) = 1$

Direct Estimation

Estimating parametric flow:

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_7 xy + \alpha_8 x^2$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y + \alpha_8 xy + \alpha_7 y^2$$

$$(\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_7 xy + \alpha_8 x^2) I_x + (....)\ I_y + I_t = 0$$

Every pixel contributes one linear equation for the 8 unknowns

Direct Estimation

Estimating 3-frame Motion:

$$u = \frac{1}{Z} s^T t + s^T [w]_x \, p$$

$$v = \frac{1}{Z} r^T t + r^T [w]_x \, p$$

Combine with: $$uI_x + vI_y + I_t = 0$$

$$\frac{1}{Z}(I_x s + I_y r)^T t + (I_x s + I_y r)^T [w]_\times \, p + I_t = 0$$

Direct Estimation

$$\frac{1}{Z}(I_x s + I_y r)^T t + (I_x s + I_y r)^T [w]_\times p + I_t = 0$$

Let

$$h = I_x s + I_y r = \begin{pmatrix} I_x \\ I_y \\ -xI_x - yI_y \end{pmatrix}$$

$$q = p \times h = \begin{pmatrix} -xyI_x - y^2 I_y - I_y \\ I_x + x^2 I_x + xyI_y \\ xI_y - yI_x \end{pmatrix}$$

Class 3 ⇒

$$\boxed{\frac{1}{Z} h^T t + w^T q + I_t = 0}$$

Direct Estimation

$$\frac{1}{Z} h^T t + w^T q + I_t = 0$$

image 1 to image 2

$$\frac{1}{Z} h^T t' + w'^T q + I'_t = 0$$

image 1 to image 3

$$I'_t h^T t - I_t h^T t' + h^T [t w'^T - t' w^T] q = 0$$

Each pixel contributes a linear equation to the 15 unknown parameters

Direct Estimation: Factorization

Let (u_{ij}, v_{ij}) be the "flow" of point i at image j (image 0 is ref frame)

$$\begin{bmatrix} u_{11} & u_{12} & \dots & u_{1m} \\ . & . & . & . \\ . & . & . & . \\ u_{n1} & u_{n2} & \dots & u_{nm} \\ v_{11} & v_{12} & \dots & v_{1m} \\ . & . & . & . \\ . & . & . & . \\ v_{n1} & v_{n2} & \dots & v_{nm} \end{bmatrix} = \begin{bmatrix} p_1 \times s_1 & (1/Z)s_1^T \\ . & . \\ . & . \\ p_n \times s_n & (1/Z)s_n^T \\ p_1 \times r_1 & (1/Z)r_1^T \\ . & . \\ . & . \\ p_n \times r_n & (1/Z)r_n^T \end{bmatrix}_{2n\times 6} \begin{bmatrix} w_1 & \dots & w_m \\ t_1 & \dots & t_m \end{bmatrix}_{6\times m}$$

$$W = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} S_x \\ S_y \end{bmatrix} M = SM$$

Direct Estimation: Factorization

$$\begin{bmatrix} I_x & I_y \end{bmatrix}_{n\times 2n} \begin{bmatrix} U \\ V \end{bmatrix}_{2n\times m} + I_t = 0$$

$$I_x = diag\,(I_{x_1}, \ldots, I_{x_n})$$

$$I_y = diag\,(I_{y_1}, \ldots, I_{y_n})$$

$$I_t = \begin{bmatrix} I_{t_1}^1 & \ldots & I_{t_1}^m \\ . & . & . \\ . & . & . \\ . & . & . \\ I_{t_n}^1 & \ldots & I_{t_n}^m \end{bmatrix}_{n\times m}$$

Direct Estimation: Factorization

Recall:

$$\begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\sum I_x I_t \\ -\sum I_y I_t \end{pmatrix}$$

⇨

$$\begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix} \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} = \begin{pmatrix} g_{ij} \\ h_{ij} \end{pmatrix}$$

Direct Estimation: Factorization

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}_{2n\times 2n} \begin{bmatrix} U \\ V \end{bmatrix}_{2n\times m} = \begin{bmatrix} G \\ H \end{bmatrix}_{2n\times m}$$

$$A = diag\ (a_1,\dots,\ a_n)$$

$$B = diag\ (b_1,\dots,\ b_n)$$

$$C = diag\ (c_1,\dots,\ c_n)$$

$$G = \begin{bmatrix} g_{11} & \cdots & g_{m1} \\ . & . & . \\ . & . & . \\ . & . & . \\ g_{n1} & \cdots & g_{mn} \end{bmatrix}_{n\times m}$$

$$H = \begin{bmatrix} h_{11} & \cdots & h_{m1} \\ . & . & . \\ . & . & . \\ . & . & . \\ h_{n1} & \cdots & h_{mn} \end{bmatrix}_{n\times m}$$

Direct Estimation: Factorization

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}_{2n\times 2n} \begin{bmatrix} U \\ V \end{bmatrix}_{2n\times m} = \begin{bmatrix} G \\ H \end{bmatrix}_{2n\times m}$$

Rank=6 Rank=6

Enforcing rank=6 constraint on the measurement matrix $\begin{bmatrix} G \\ H \end{bmatrix}$ removes errors in a least-squares sense.

Direct Estimation: Factorization

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}_{2n\times 2n} \begin{bmatrix} U \\ V \end{bmatrix}_{2n\times m} = \begin{bmatrix} G \\ H \end{bmatrix}_{2n\times m}$$

$$\Rightarrow \quad \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}^{\#} \begin{bmatrix} G \\ H \end{bmatrix}$$

Once U,V are recovered, one can solve for S,M as before.