Multi-linear Systems and Invariant Theory

in the Context of Computer Vision and Graphics

Class 2: Homography Tensors

CS329
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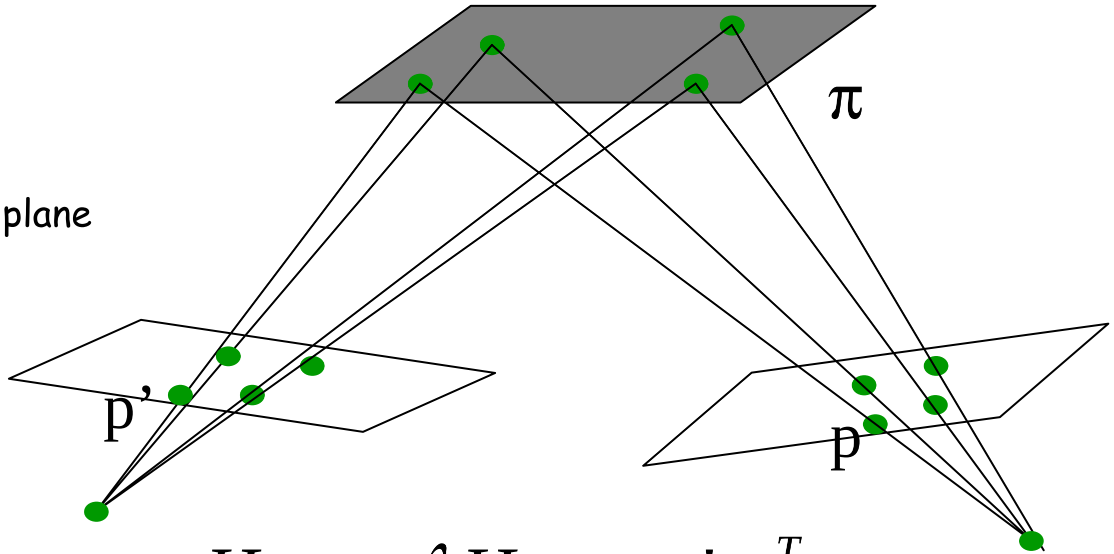
Material We Will Cover Today

- 2D-2D mapping of a "dynamic" point configuration
- Primer on Tensor Products and Covariant-Contravariant conventions
- Homography Tensors and their properties

Homography Matrix

$$p' \cong H_\pi p$$

4 points make a basis for the projective plane

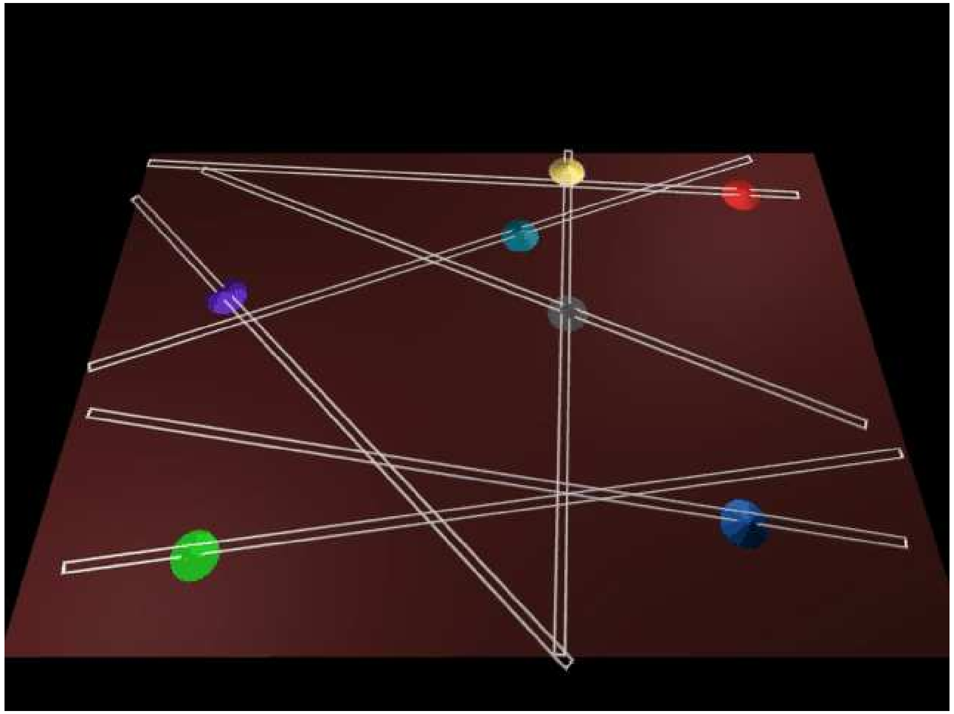


$$H_\pi = \lambda H_\infty + e' n^T$$

H_π Stands for the family of 2D projective transformations between two fixed images induced by a plane in space

$P^2 \otimes P^2 \rightarrow P^2$ Mapping of the dynamic projective plane onto itself

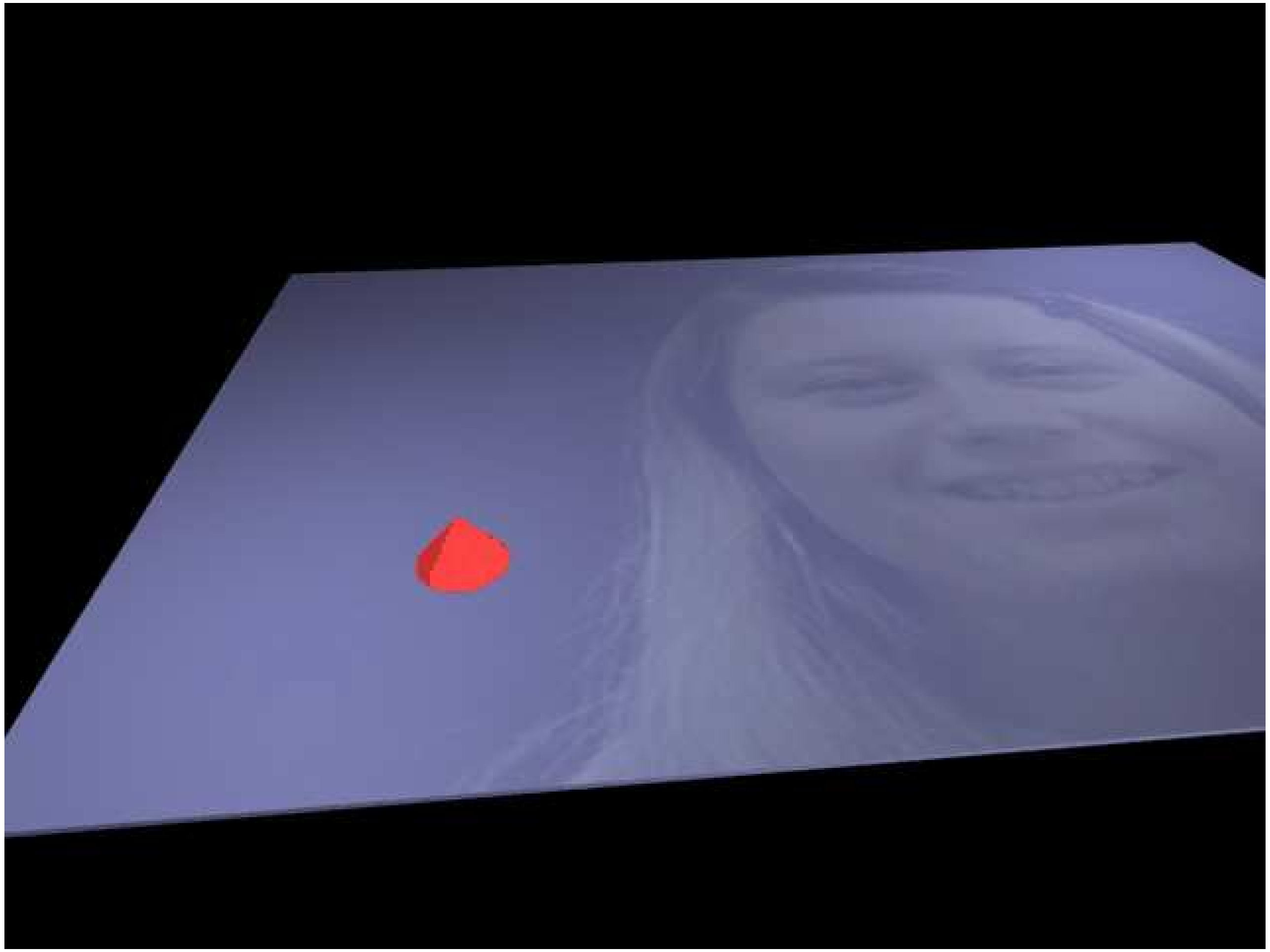
(movie)



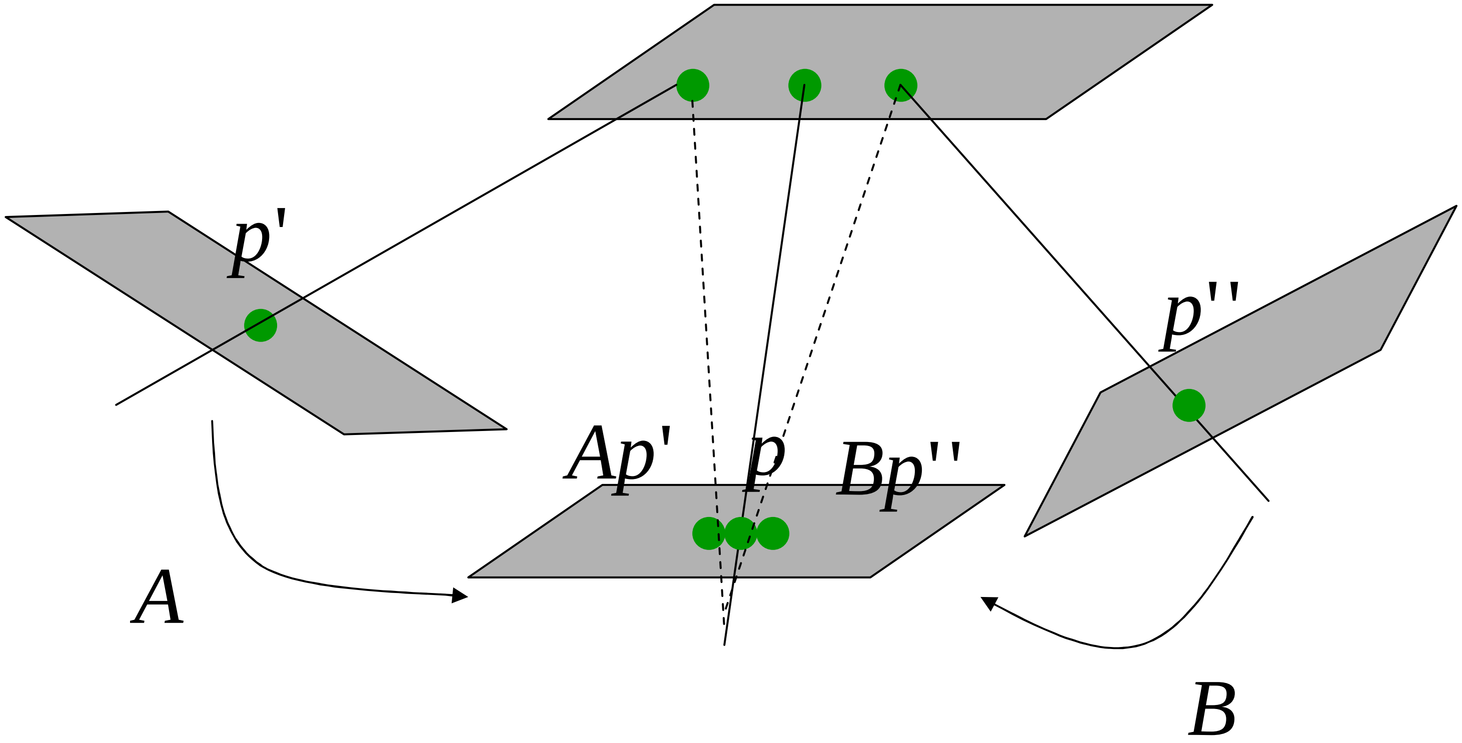
Points are moving along straight-line trajectories
while camera changes position

3 snapshots of a linearly moving point

(movie)



A,B are unknown



$$rank\begin{bmatrix} p & Ap' & Bp'' \end{bmatrix} = 2$$

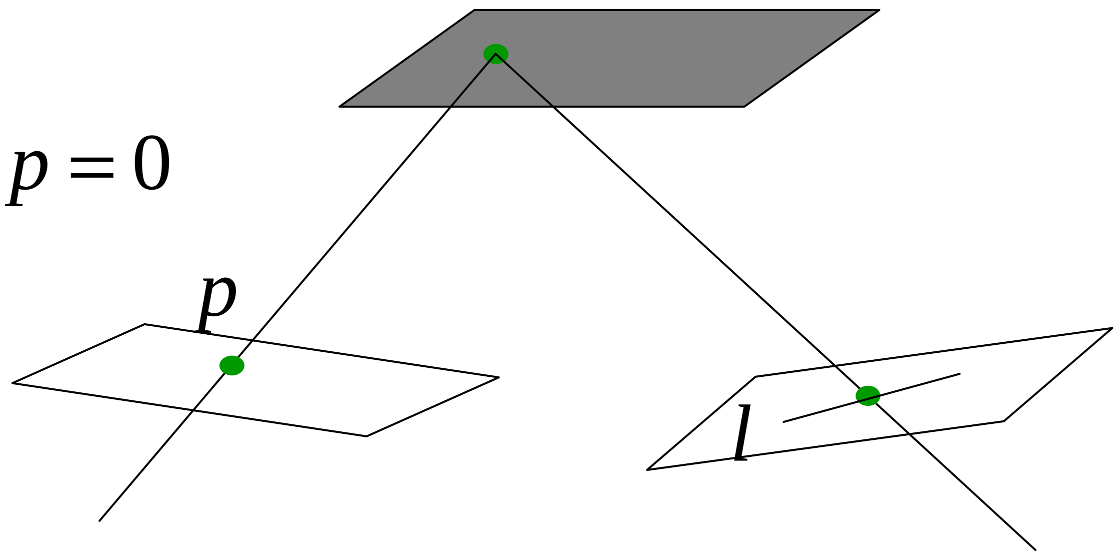
$$\Longrightarrow \quad p^T (Ap' \times Bp'') = 0$$

Multilinear relation between p,p',p'' and A,B

Tensor Products

- Combine Linear transformations in a way that "makes sense"
- Coefficients of a multilinear form are arranged as a mapping

$$l^T H_\pi p = 0$$



Tensor Products

- Combine Linear transformations in a way that "makes sense"
- Coefficients of a multilinear form are arranged as a mapping
- Fundamental in Group theory: construct new representations by tensor product of old representations (chemistry, physics)
- Appears under different formalisms:
    - $V \otimes W$
    - $T_i^{\ jk}$
    - Grassman Analysis
    - Dual Algebra, Extensors, Geometric Algebra…

$$f : V \times W \to U$$ U,V,W are vector spaces

is bilinear if

$$f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$$

$$f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$$

$$f(\alpha v, w) = f(v, \alpha w) = \alpha f(v, w)$$

Example: dimV=dimW=dimU=3, the regular “cross product”

$$v \times w = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ -v_1 w_3 + v_3 w_1 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = u$$

Tensor Product: Definition

The tensor product of two vector spaces V,W is a vector space

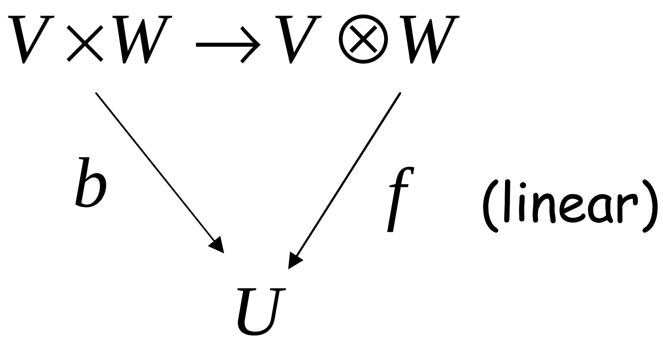
$V \otimes W$ equipped with a bilinear map

$$V \times W \to V \otimes W$$

Which is universal: for any bilinear map $b : V \times W \to U$

There is *unique linear* map f from $V \otimes W$ to U that takes

$v \otimes w$ to $b(v, w)$



Constructing Tensor Products

$e_1, \ldots, e_n$ be a basis for V

$f_1, \ldots, f_m$ be a basis for W

$e_i \otimes f_j$ is a basis for $V \otimes W$ $\dim(V \otimes W) = nm$

$$(ae_i + be_j) \otimes f_k = ae_i \otimes f_k + be_j \otimes f_k$$

$$e_k \otimes (af_i + bf_j) = ae_k \otimes f_i + be_k \otimes f_j$$

$$(ae_i) \otimes f_j = e_i \otimes (af_j) = a(e_i \otimes f_j)$$

Combining Linear Maps

$A: V \to V'$ $\qquad A \in R^{n \times n'}$

$B: W \to W'$ $\qquad B \in R^{m \times m'}$

$$(A \otimes B)(e_i \otimes f_j) = Ae_i \otimes Bf_j$$

(consequence of universal property)

The map sending $(v, w) \in V \times W$ to $Av \otimes Bw$ is bilinear

⟹ There exists a unique linear map $V \otimes W \to V' \otimes W'$

denoted by $A \otimes B$ such that

$$(A \otimes B)(v \otimes w) = Av \otimes Bw$$

Class 2 ⟹ $$(A' \otimes B')(Av \otimes Bw) = AA'v \otimes BB'w$$

Example

What does $A \otimes B$ look like?

$\dim(V, V', W, W') = 2$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2$ are the 4 basis elements

$$Ae_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = ae_1 + ce_2$$

$$Ae_2 = be_1 + de_2$$

$$(A \otimes I)(e_1 \otimes f_1) = Ae_1 \otimes f_1 = (ae_1 + ce_2) \otimes f_1 = a(e_1 \otimes f_1) + c(e_2 \otimes f_1)$$

$$(A \otimes I)(e_1 \otimes f_2) = a(e_1 \otimes f_2) + c(e_2 \otimes f_2)$$

$$(A \otimes I)(e_2 \otimes f_1) = b(e_1 \otimes f_1) + d(e_2 \otimes f_1)$$

$$(A \otimes I)(e_2 \otimes f_2) = b(e_1 \otimes f_2) + d(e_2 \otimes f_2)$$

$$\Longrightarrow \quad A \otimes I = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}$$

$$Bf_1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \alpha f_1 + \gamma f_2$$

$$Bf_2 = \beta e_1 + \delta e_2$$

$$(I \otimes B)(e_1 \otimes f_1) = e_1 \otimes Bf_1 = e_1 \otimes (\alpha f_1 + \gamma f_2) = \alpha(e_1 \otimes f_1) + \gamma(e_1 \otimes f_2)$$

$$(I \otimes B)(e_1 \otimes f_2) = \beta(e_1 \otimes f_1) + \delta(e_1 \otimes f_2)$$

$$(I \otimes B)(e_2 \otimes f_1) = \alpha(e_2 \otimes f_1) + \gamma(e_2 \otimes f_2)$$

$$(I \otimes B)(e_2 \otimes f_2) = \beta(e_2 \otimes f_1) + \delta(e_2 \otimes f_2)$$

$$\Rightarrow \quad I \otimes B = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & \delta \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

Recall:

$$(A'\otimes B')(A\otimes B)(v\otimes w) = AA'v\otimes BB'w$$

$\Longrightarrow$

$$(I\otimes B)(A\otimes I)(e_i\otimes f_j) = (A\otimes B)(e_i\otimes f_j) = Ae_i\otimes Bf_j$$

$$A\otimes B = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} aB & bB \\ cB & dB \end{bmatrix}$$

$$\boxed{(A\otimes B)_{ijkl} = a_{ij}b_{kl}}$$

"outer product"

To conclude, we see that the "proper" combination of two matrices A,B (the tensor product of A,B) is a new matrix whose entries are:

$$a_{ij}b_{kl} = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ . & . & . & . \\ . & . & . & . \\ a_{n1}B & a_{n2}B & \ldots & a_{nn}B \end{bmatrix}_{nm\times nm}$$

This brings us to the *index* notations, described next.

Index Notations

Goal: represent the operations of inner-product and outer-product

A vector has super-script running index when it represents a *point*

A vector has subscript running index when it represents a *hyperplane*

Example:

$$p^i = (p^1, p^2, p^3)$$ Represents a *point* in the projective plane

$$s_j = (s_1, s_2, s_3)$$ Represents a *line* in the projective plane

An *outer-product*:

$u_i v_j$ is an object (2-valence tensor) whose entries are

$$u_1 v_1, \ldots, u_1 v_m, \ldots, u_n v_1, \ldots, u_n v_m$$

Note: this is the outer-product of two vectors:

$$uv^T = \begin{bmatrix} u_1 v_1 & \ldots & u_1 v_m \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_n v_1 & \ldots & u_n v_m \end{bmatrix}_{n \times m}$$

(rank-1 matrix)

$$a_{ij} = c_i c'_j + d_i d'_j + \ldots + x_i x'_j$$

A general 2-valence tensor is a sum of rank-1 2-valence tensors

Likewise,

$$u_i v^j$$

$$u^i v^j$$

$$u^i v_j$$

Are outer-products consisting of the same elements, but as a mapping carry each a different meaning (described later).

These are also called *mixed* tensors, where the super-script is called *contra-variant* index and the *subscript* is called covariant index.

The *inner-product* (contraction):

Summation rule: same index in contravariant and covariant positions are summed over. This is sometimes called the "Einstein summation convention".

$$u_j v^j = u_1 v^1 + u_2 v^2 + \ldots + u_n v^n$$

The *inner-product* (contraction):

$$a_i^{\ j} u^i = a_1^{\ j} u^1 + a_2^{\ j} u^2 + \ldots + a_n^{\ j} u^n = v^j$$

Note: this is the familiar matrix-vectors multiplication: $Au = v$
where the super-script j runs over the rows of the matrix

Note: the 2-valence tensor a_i^j maps points to points

Likewise,

$$a_i^{\ j} u_j = v_i \qquad \text{Maps hyperplanes (lines in 2D) to hyperplanes}$$

Note: this is equivalent to $A^T u = v$

We have seen in the past that if $Hp \cong p'$ is a homography

Then $H^{-T} l \cong l'$ maps lines from view 1 to view 2

Let p_1, p_2, p_3 Colinear points, i.e.

$l^T p_i = 0$ ⟹ $l^T H^{-1} H p_i = 0$

⟹ the points Hp_i lie on the line $H^{-T} l$

With the index notations we get this property immediately!

The complete list:

$$a_i^{\ j} u^i = v^j$$

Maps points to points

$$a_i^{\ j} u_j = v_i$$

Maps hyperplanes (lines in 2D) to hyperplanes

$$a_{ij} u^i = v_j$$

Maps points to hyperplanes

$$a^{ij} u_i = v^j$$

Maps hyperplanes to points

More Examples:

runs over the rows

$$a_j^k b_i^j = c_i^k$$

This is the matrix product $AB = C$

runs over the columns

$$u^i v_j H_i^{jk}$$

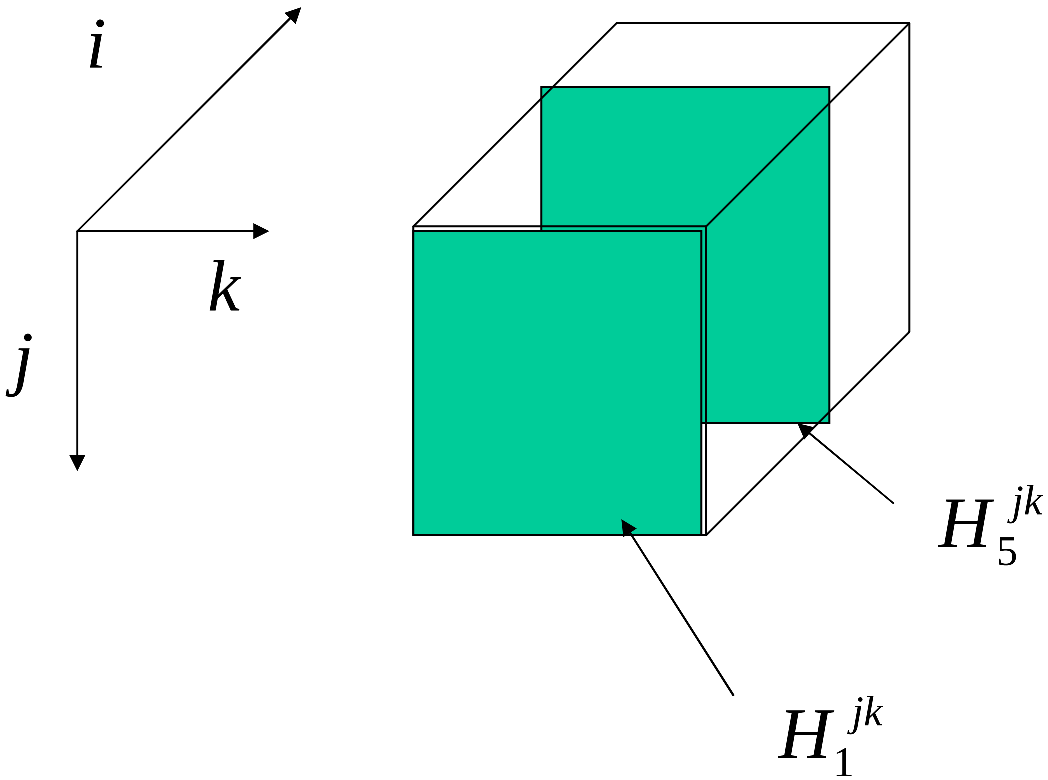
Must be a point

$\Rightarrow$ H_i^{jk} Takes a point in first frame, a hyperplane in the second frame and produces a point in the third frame

$$u^i H_i^{jk}$$

Must be a matrix (2-valence tensor)

if u=(1,0,0…0) then this is a *slice* H_1^{jk} of the tensor.

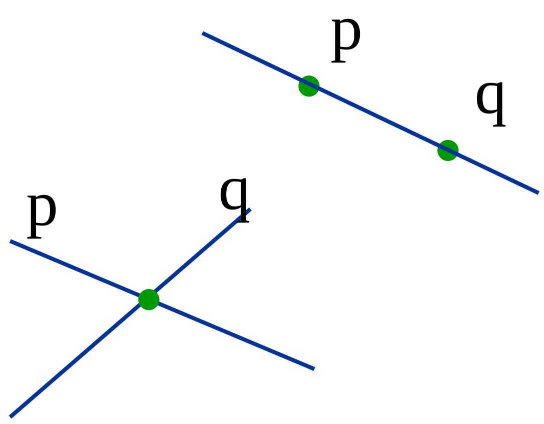

i
k
j
H_5^{jk}
H_1^{jk}

The Cross-product Tensor

$$s = p \times q = \begin{bmatrix} \det\begin{bmatrix} p_2 & q_2 \\ p_3 & q_3 \end{bmatrix} \\ -\det\begin{bmatrix} p_1 & q_1 \\ p_3 & q_3 \end{bmatrix} \\ \det\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} \end{bmatrix}$$

$$\varepsilon_{ijk} p^i q^j = s_k$$

$$\varepsilon^{ijk} p_i q_j = s^k$$



$$u \times v = [u]_\times v$$

$$[u]_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

The cross-product tensor is defined such that $\varepsilon^{ijk} u_i$

Produces the matrix $[u]_\times$ i.e., the entries are 1,-1,0

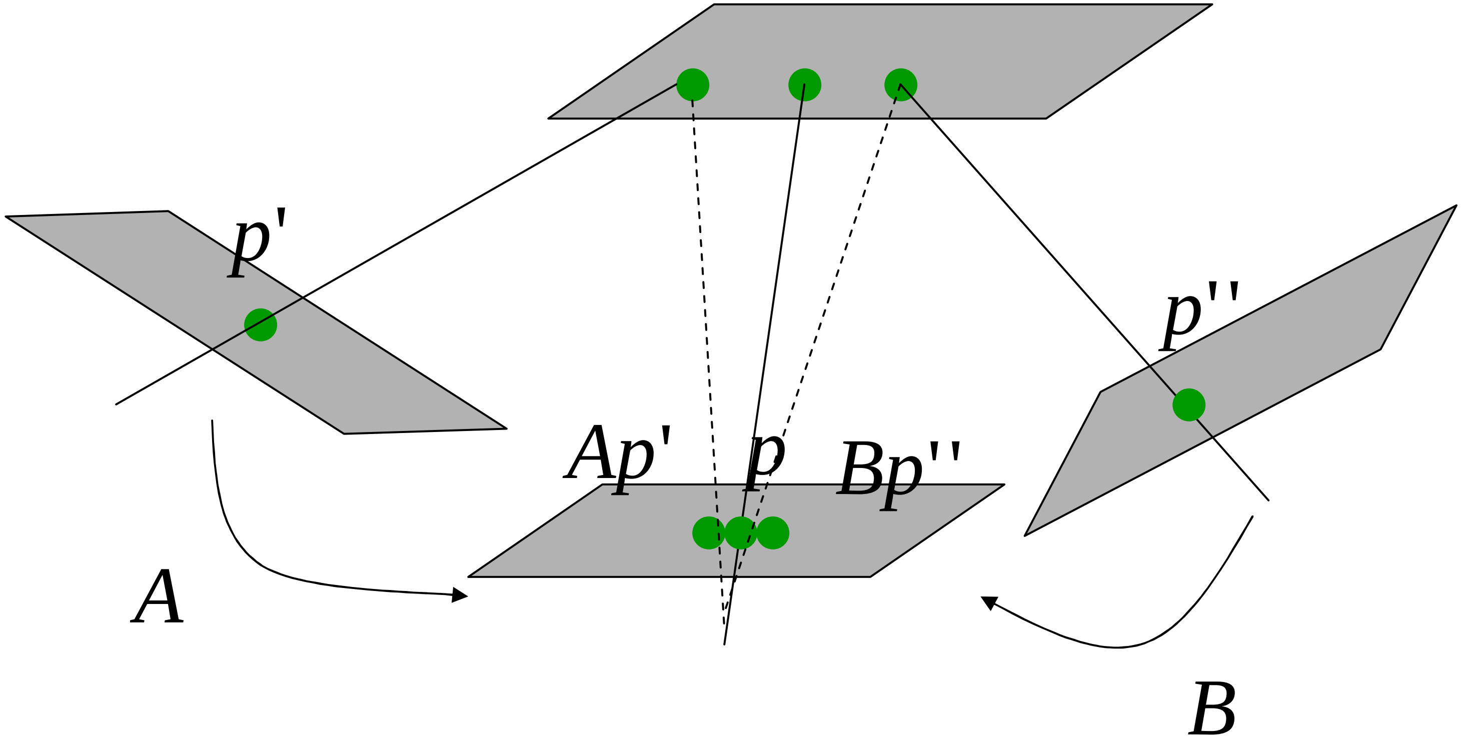
$$\varepsilon^{1jk} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \varepsilon^{2jk} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \varepsilon^{3jk} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$u_i \varepsilon^{ijk} = u_1 \varepsilon^{1jk} + u_2 \varepsilon^{2jk} + u_3 \varepsilon^{3jk} = [u]_\times$$

$$[u]_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

End of Primer on Tensors

A,B are unknown



$$rank\begin{bmatrix} p & Ap' & Bp'' \end{bmatrix} = 2$$

$$\Longrightarrow \quad p^T(Ap' \times Bp'') = 0$$

Multilinear relation between p,p',p'' and A,B

$$p^T (Ap' \times Bp'') = 0$$

Let index i run over view 1, index j over view 2 and index k over view 3

$$p^i \varepsilon_{inu} (a_j^{\ n} p'^j)(b_k^{\ u} p''^k) = 0$$

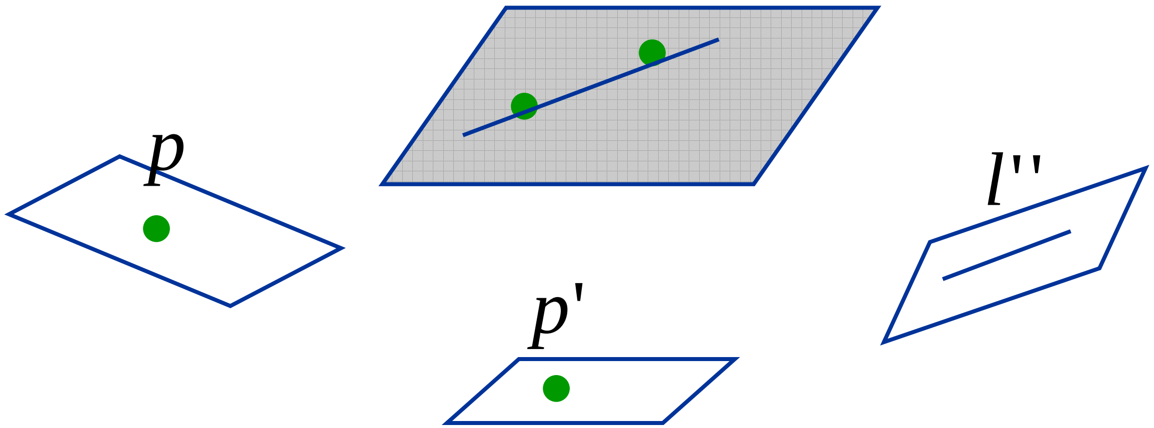
The position of symbols does not matter (only the indices)

$$p^i p'^j p''^k (\varepsilon_{inu} a_j^{\ n} b_k^{\ u}) = 0$$

$$\Longrightarrow \quad p^i p'^j p''^k H_{ijk} = 0$$

H_{ijk} as a mapping and its slices

$$p^i p'^j H_{ijk} = l''_k$$

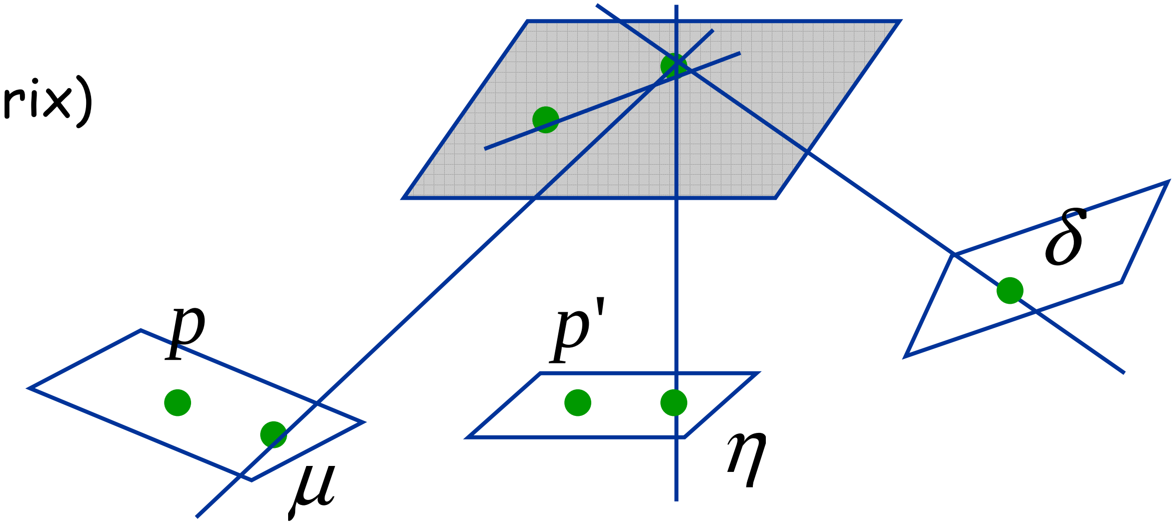


From index structure l'' must be a line. Because $p^i p'^j p''^k H_{ijk} = 0$

For every point along the straight line trajectory determined by p,p' then the line l'' must the projection of that trajectory.

$$\delta^k H_{ijk} = E_{ij}$$ (a matrix)

$$H_{ijk} = \varepsilon_{inu} a_j{}^n b_k{}^u$$



$$\delta^k H_{ijk} = \varepsilon_{inu} a_j{}^n (b_k{}^u \delta^k) = [B\delta]_\times A$$

$$\Longrightarrow \quad E = [\mu]_x A \qquad \mu = B\delta$$

$$E\eta = [\mu]_x A\eta \cong [\mu]_x \mu = 0$$

$$E^T \mu = -A^T [\mu]_x \mu = 0$$

$$p^T E p' = 0$$ For all pairs p,p' on matching lines through the fixed points μ and η

$$E = [\mu]_x A$$

$$E^T A = -A^T [\mu]_x A$$

$$A^T E = A^T [\mu]_x A$$

$$\Longrightarrow \quad A^T E + E^T A = 0$$

Given E we obtain 6 linear equations to solve for A

With 2 slices,

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^k H_{ijk} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix}^k H_{ijk}$$ one can solve for A

Estimation of H_{ijk}

- 26 matching triplets p,p’,p’’ arising from dynamic points provide A unique solution for the dual Htensor (each triplet provides one linear constraint).

- The 26 points must lie on *at least* 4 lines (in general position), where no more than 8 points on the first line, no more than 7 points on the second line, 6 on the third, and 5 on the fourth. *(proof by principle of duality with Htensor).*

Labeled Static Points

$$rank[p \quad Ap' \quad Bp''] = 1$$

$$p^i p'^j H_{ijk} = 0 \qquad p^i p''^k H_{ijk} = 0 \qquad p'^j p''^k H_{ijk} = 0$$

⇨ 9 linear constraints on H

$$p^i p'^j e^k H_{ijk} = 0, \forall e$$
$$p^i p''^k e^j H_{ijk} = 0, \forall e$$
$$p'^j p''^k e^i H_{ijk} = 0, \forall e$$

But $p^i p'^j p''^k H_{ijk} = 0$ Appears 3 times!

⇨ 7 linearly independent constraints on H

⇨ 4 (labeled) static points are sufficient for solving for H

Unlabeled Static Points

What if all the measurements arise from static points without Prior knowledge that they are static? (unlabeled static)

$$p^i p'^j p''^k H_{ijk} = 0 \quad \Longrightarrow \quad G^{ijk} H_{ijk} = 0$$

It is sufficient to consider A=B=I $\Longrightarrow$ $G^{ijk} = p^i p^j p^k$

G^{ijk} is a symmetric tensor, i.e. contains only 10 different groups

111,222,333,112,113,221,223,331,332,123

up to permutations. $rank(v_1 \otimes v_2 \otimes v_3 \mid rankspan(v_1, v_2, v_3) = 1) = \binom{n+3-1}{3}$

$\Longrightarrow$ One needs at least 16 dynamic points in an unlabeled set

Mixed Labeled and Unlabeled Static Points

- 3 of the 7 constraints provided by a labeled static live in the 10'th dimensional subspace of unlabeled static points.

$$\Downarrow$$

- If we have $0 \le x \le 4$ labeled static points, then we need

$16 - 4x$ dynamic points

Partly segmented scene

Known static	unknown	moving required
0	26	16
1	19	12
2	12	8
3	5	4
4	0	0