Multi-linear Systems and Invariant Theory
in the Context of Computer Vision and Graphics

CS329
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Material We Will Cover Today

- The structure of 3D->2D projection matrix
- The homography matrix
- A primer on projective geometry of the plane
The Structure of a Projection Matrix

\[ P_w = \begin{pmatrix} U \\ V \\ W \end{pmatrix} \]

\[ \overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{Ut} + \overrightarrow{Vn} + \overrightarrow{Wb} \]

\[ R = \begin{bmatrix} t & \vec{n} & \vec{b} \end{bmatrix} \]

\[ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} U \\ V \\ W \end{pmatrix} + T \]
The Structure of a Projection Matrix

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

\[
x = f \frac{X}{Z} + x_0
\]

\[
y = f \frac{Y}{Z} + y_0
\]
The Structure of a Projection Matrix

\[ x = f \frac{X}{Z} + x_0 \]

\[ y = f \frac{Y}{Z} + y_0 \]

\[
\begin{pmatrix}
  x \\
  y \\
  1
\end{pmatrix}
= \begin{bmatrix}
  f & 0 & x_0 \\
  0 & f & y_0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
  X \\
  Y \\
  Z
\end{pmatrix}
= K(RP_w + T) = K[R;T]
\]

\[ p \cong M_{3\times4} P \]
The Structure of a Projection Matrix

Generally,

\[
K = \begin{bmatrix}
    f_x & f_x \frac{\cos \theta}{\sin \theta} & x_0 \\
    0 & \frac{f_y}{\sin \theta} & y_0 \\
    0 & 0 & 1
\end{bmatrix}
\]

\((x_0, y_0)\) is called the “principle point”

\(s = f_x \frac{\cos \theta}{\sin \theta}\) is called the “skew”

\(\frac{f_y}{f_x}\) is “aspect ratio”
The Camera Center

\[ p \equiv M_{3 \times 4} P \]

\[ M \] has rank=3, thus \( \exists c \) such that \( Mc = 0 \)

\[ K[R;T]c = 0 \Rightarrow [R;T]c = 0 \Rightarrow c = \begin{pmatrix} -R^T T \\ 1 \end{pmatrix} \]

Why is \( c \) the camera center?
The Camera Center

Why is $c$ the camera center?

Consider the “optical ray” $Q(\lambda) = \lambda P + (1 - \lambda)c$

$$MQ(\lambda) = \lambda MP \cong MP \cong p$$

All points along the line $Q(\lambda)$ are mapped to the same point $p$.

$Q(\lambda)$ is a ray through the camera center.
The Epipolar Points

c = M'e

c' = M'c

e = Mc'
Choice of Canonical Frame

\[ p \cong MP = MWW^{-1}P \]

\[ p' \cong M'P = M'WW^{-1}P \]

\[ W^{-1}P \] is the new world coordinate frame

We have 15 degrees of freedom (16 upto scale)

Choose \( W \) such that \( MW = [I;0] \)
Choice of Canonical Frame

Let \( M = \begin{bmatrix} M^* ; m \end{bmatrix} \)

\[
MW = \begin{bmatrix} M^* ; m \end{bmatrix} \begin{bmatrix} M^{-1} - (\overline{Mm})n^T & -(1/\lambda)\overline{M^{-1}}m \\ n^T & 1/\lambda \end{bmatrix} \\
= \begin{bmatrix} I - mn^T + mn^T ; -(1/\lambda)m + (1/\lambda)m \end{bmatrix} \\
= \begin{bmatrix} I ; 0 \end{bmatrix}
\]

We are left with 4 degrees of freedom (upto scale): \((n^T, \lambda)\)
Choice of Canonical Frame

\[ p \equiv [I;0] \bar{P} \]

\[ p' \equiv [H;e'] P \]

\[ p \equiv [I;0] \begin{bmatrix} I & 0 \\ n^T & 1 / \lambda \end{bmatrix} \begin{bmatrix} I & 0 \\ -\lambda n^T & \lambda \end{bmatrix} \bar{P} \]

\[ \begin{bmatrix} I & 0 \\ -\lambda n^T & \lambda \end{bmatrix} \bar{P} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = P \]
Choice of Canonical Frame

\[ p' \equiv [H; e'] \begin{pmatrix} p \\ \mu' \end{pmatrix} = [H; e'] \begin{bmatrix} I & 0 \\ n^T & 1 / \lambda \end{bmatrix} \begin{pmatrix} p \\ \mu \end{pmatrix} \]

\[ [H; e'] \begin{bmatrix} I & 0 \\ n^T & 1 / \lambda \end{bmatrix} = [H + e'n^T; (1/\lambda)e'] \]

\[ \cong [\lambda H + e'n^T; e'] \]

\[ p' \equiv [\lambda H + e'n^T; e'] \begin{pmatrix} p \\ \mu \end{pmatrix} \]

where \((n^T, \lambda)\) are free variables
Projection Matrices

Let $p^j_i$ be the image of point $P_i$ at frame number $j$

$$p^0_i \equiv [I;0]P_i$$

$$p^j_i \equiv [\lambda H_j + e_j n^T; e_j]P_i$$

where $(n^T, \lambda)$ are free variables

$p^j_i$ are known
Family of Homography Matrices

\[ p^j_i \equiv [\lambda H^j + e_j n^T; e_j] P_i \]

\[ H_\pi = \lambda H + e' n^T \]

\( H_\pi \) stands for the family of 2D projective transformations between two fixed images induced by a plane in space.

The remainder of this class is about making the above statement intelligible.
Family of Homography Matrices

Recall,

\[ p \cong K[R; T] \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix} \]

3D->2D from Euclidean world frame to image

\[
\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} U \\ V \\ W \end{pmatrix} + T
\]

world frame to first camera frame

Let \( K, K' \) be the internal parameters of camera 1,2 and choose canonical frame in which \( R=I \) and \( T=0 \) for first camera.

\[
p \cong [K ; 0] P
\]

\[
p' \cong K'[R ; t] P
\]
Family of Homography Matrices

\[ p \cong [K; 0]P \]

\[ p' \cong K'[R \ t]P \]

\[ p = \frac{1}{Z} K \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \]

Recall that 3rd row of K is (0,0,1)

\[ p' \cong K'[R \ t] \begin{pmatrix} ZK^{-1}p \\ 1 \end{pmatrix} = ZK'RK^{-1}p + K't \]

\[ p' \cong K'RK^{-1}p + \frac{1}{Z} K't \]
Family of Homography Matrices

\[ p' \equiv K'RK^{-1} p + \frac{1}{Z} K't \]

Assume \( \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \) are on a planar surface

\[ \overline{n}^T \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = d \]

\[ \overline{n}^T (ZK^{-1} p) = d \]

\[ p' \equiv K'RK^{-1} p + \frac{1}{Z} K't \left( \frac{Z\overline{n}^T K^{-1} p}{d} \right) \]
Family of Homography Matrices

\[
p' \equiv K'RK^{-1}p + \frac{1}{Z}K't\left(\frac{Zn^TK^{-1}p}{d}\right)
\]

\[
p' \equiv (K'RK^{-1} + \frac{1}{d}K'tn^TK^{-1})p
\]

Let \( e' = K't, \quad n^T = n^TK^{-1} \) and \( H_\infty = K'RK^{-1} \)

\[
p' \equiv (H_\infty + \frac{1}{d}e'n^T)p
\]

Image-to-image mapping \( p \rightarrow p' \) where the matching pair \( p, p' \) are induced by a planar surface.
Family of Homography Matrices

\[ H_\pi \cong H_\infty + \frac{1}{d} e' n^T \]

when \( d \to \infty \)

\[ H_\pi \to H_\infty \cong K' RK^{-1} \]

\( K' RK^{-1} \) is the homography matrix induced by the plane at infinity
Family of Homography Matrices

\[ H_\pi \equiv H_\infty + \frac{1}{d} e' n^T \]

\[ e' = K' t \quad \text{is the epipole in the second image} \]

\[ M = [K; 0] \]

\[ M' = K' [R \ t] \]

\[ c^T = (0,0,0,1) \]

\[ e' = M' c = K' t \]
Relationship Between two Homography Matrices

\[ H_{\pi} \cong H_{\infty} + \frac{1}{d} e' n^T \]

\[ H_{\pi_1} \cong \lambda H_{\pi_2} + e'l^T \]

\[ H_{\pi_1} u \cong H_{\pi_2} u \quad \forall u \in l \]

\[ u^T l = 0 \]

\[ l \] is the projection of \( L = \pi_1 \wedge \pi_2 \) onto the first image
Estimating the Homography Matrix

How many matching points?

\[ x' = \frac{h_1^T p}{h_3^T p} \]

\[ y' = \frac{h_2^T p}{h_3^T p} \]

\[ p' \cong H_\pi p \]

4 points make a basis for the projective plane
Projective Geometry of the Plane

\[ ax + by + c = 0 \]  
Equation of a line in the 2D plane

The line is represented by the vector \( l = (a, b, c)^T \) and \( p^T l = 0 \)

\[ p = (x, y, 1)^T \]

Correspondence between lines and vectors are not 1-1 because \( (\lambda a, \lambda b, \lambda c)^T \) represents the same line \( p^T (\lambda l) = 0, \forall \lambda \neq 0 \)

The vector \( (0,0,0)^T \) does not represent any line.

Two vectors differing by a scale factor are equivalent. This equivalence class is called homogenous vector. Any vector \( (a, b, c)^T \) is a representation of the equivalence class.
Projective Geometry of the Plane

A point \((x, y)\) lies on the line (coincident with) which is represented by \(l = (a, b, c)^T\) iff \(p^T l = 0\)

But also \((\lambda p)^T l = 0\)

\[\begin{align*}
(I) & \quad (\lambda x, \lambda y, \lambda)^T, \forall \lambda \neq 0 \quad \text{represents the point } (x, y) \\
(II) & \quad (x_1, x_2, x_3)^T \quad \text{represents the point } \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right) \\
& \quad \text{The vector } (0,0,0)^T \text{ does represent any point.}
\end{align*}\]

Points and lines are dual to each other (only in the 2D case!).
Projective Geometry of the Plane

\[ p \equiv r \times s \]

\[ p^T r = (r \times s)^T r = 0 \]

\[ p^T s = (r \times s)^T s = 0 \]

**Note:** \((a \times b)^T c = \det(a, b, c)\)

\[ l \equiv p \times q \]
Lines and Points at Infinity

Consider lines \( r = (a, b, c)^T \) and \( s = (a, b, c')^T \):

\[
\begin{bmatrix}
bc' - cb \\
ac - ac' \\
0
\end{bmatrix} = (c' - c)
\begin{bmatrix}
b \\
0
\end{bmatrix} \equiv
\begin{bmatrix}
b \\
-a \\
0
\end{bmatrix}
\]

which represents the point \((b, -a, 0)\) with infinitely large coordinates.

All meet at the same point \((b, -a, 0)\).
Lines and Points at Infinity

The points \((x_1, x_2, 0)^T, \forall x_1, x_2\) lie on a line

\[
(x_1, x_2, 0)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0
\]

The line \(l_\infty = (0, 0, 1)^T\) is called the line at infinity.

The points \((x_1, x_2, 0)^T, \forall x_1, x_2\) are called ideal points.

A line \((a, b, \lambda)^T\) meets \(l_\infty = (0, 0, 1)^T\) at \((b, -a, 0)^T\)

(which is the direction of the line)
A Model of the Projective Plane

Points are represented as lines (rays) through the origin
Lines are represented as planes through the origin

\( l_\infty = (0,0,1)^T \)

is the plane \( x_1 x_2 \)

\( (x_1, x_2, 0)^T \)

ideal point

\( (\lambda x_1, \lambda x_2, \lambda x_3)^T, \lambda \neq 0 \)
A Model of the Projective Plane

\[
P^n = \{ [x_1, \ldots, x_n] \neq [0, \ldots, 0] : [x_1, \ldots, x_n] = [\lambda x_1, \ldots, \lambda x_n], \ \forall \lambda \neq 0 \}
\]

\[= \{ \text{lines through the origin in } \mathbb{R}^{n+1} \} \]

\[= \{ \text{1-dim subspaces of } \mathbb{R}^{n+1} \} \]
Projective Transformations in $P^2$

The study of properties of the projective plane that are invariant under a group of transformations.

**Projectivity:** $h : P^2 \rightarrow P^2$

that maps lines to lines (i.e. preserves colinearity)

Any invertible 3x3 matrix is a Projectivity:

Let $p_1, p_2, p_3$ Colinear points, i.e.

$l^T p_i = 0 \quad \rightarrow \quad l^T H^{-1} H p_i = 0$

$\Rightarrow$ the points $Hp_i$ lie on the line $H^{-T} l$

$H$ is called homography, colineation $H^{-T}$ is the dual.
Projective Transformations in $P^2$

A composition of perspectivities from a plane $\pi$ to other planes and back to $\pi$ is a projectivity. Every projectivity can be represented in this way.
Projective Transformations in $\mathbb{P}^2$

Example, a prespectivity in 1D:
Lines adjoining matching points are concurrent

Lines adjoining matching points $(a,a'),(b,b'),(c,c')$ are not concurrent
Projective Transformations in $\mathbb{P}^2$

$L_\infty = (0,0,1)^T$ is not invariant under $H$:

Points on $L_\infty$ are $(x_1, x_2, 0)^T$

\[ H \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 h_1 + x_2 h_2 = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} \]

$x'_3$ is not necessarily 0

Parallel lines do not remain parallel!

$L_\infty$ is mapped to $H^{-T} L_\infty$
A **Simplex** in $\mathbb{R}^{n+1}$ is a set of $n+2$ points such that no subset of $n+1$ of them lie on a hyperplane (linearly dependent).

In $\mathbb{P}^2$ a Simplex is 4 points

Theorem: there is a unique colineation between any two Simplexes
Why do we need 4 points

**Invariants** are measurements that remain fixed under colineations

\[
\text{# of independent invariants} = \text{# d.o.f of configuration} - \text{# d.o.f of trans.}
\]

Ex: 1D case \( p' \cong H_{2 \times 2} p \) \( \rightarrow \) H has 3 d.o.f

A point in 1D is represented by 1 parameter.

4 points we have: 4-3=1 invariant (cross ratio)

2D case: H has 8 d.o.f, a point has 2 d.o.f thus 5 points induce 2 invariants
Why do we need 4 points

The cross-ratio of 4 points:

\[ \alpha = \frac{ab}{ac} \cdot \frac{cd}{bd} \]

24 permutations of the 4 points forming 6 groups:

\[ \alpha, \frac{1}{\alpha}, 1 - \alpha, \frac{\alpha - 1}{\alpha}, \frac{\alpha}{\alpha - 1}, \frac{1}{1 - \alpha} \]
Why do we need 4 points

5 points gives us 10 d.o.f, thus 10-8=2 invariants which represent 2D space. $x, y, z, u$ are the 4 basis points (simplex).

$$\alpha = \langle z, u_x, p_x, x \rangle$$

$$\beta = \langle z, u_y, p_y, y \rangle$$

$p_x, p_y$ are determined uniquely by $\alpha, \beta$.

Point of intersection is preserved under projectivity (exercise).

$p_x, p_y$ uniquely determined $p$. 