# Multi-linear Systems and Invariant Theory

# in the Context of Computer Vision and Graphics

CS329
Stanford University

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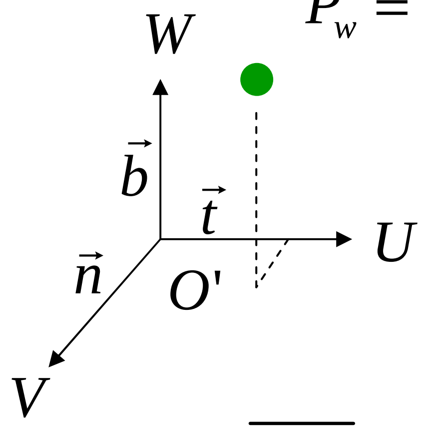
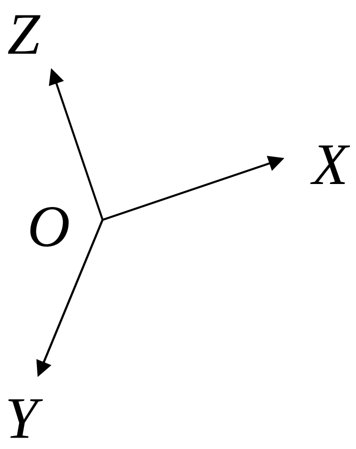
# Material We Will Cover Today

- The structure of 3D->2D projection matrix
- The homography matrix
- A primer on projective geometry of the plane

# The Structure of a Projection Matrix

$$P_w = \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$



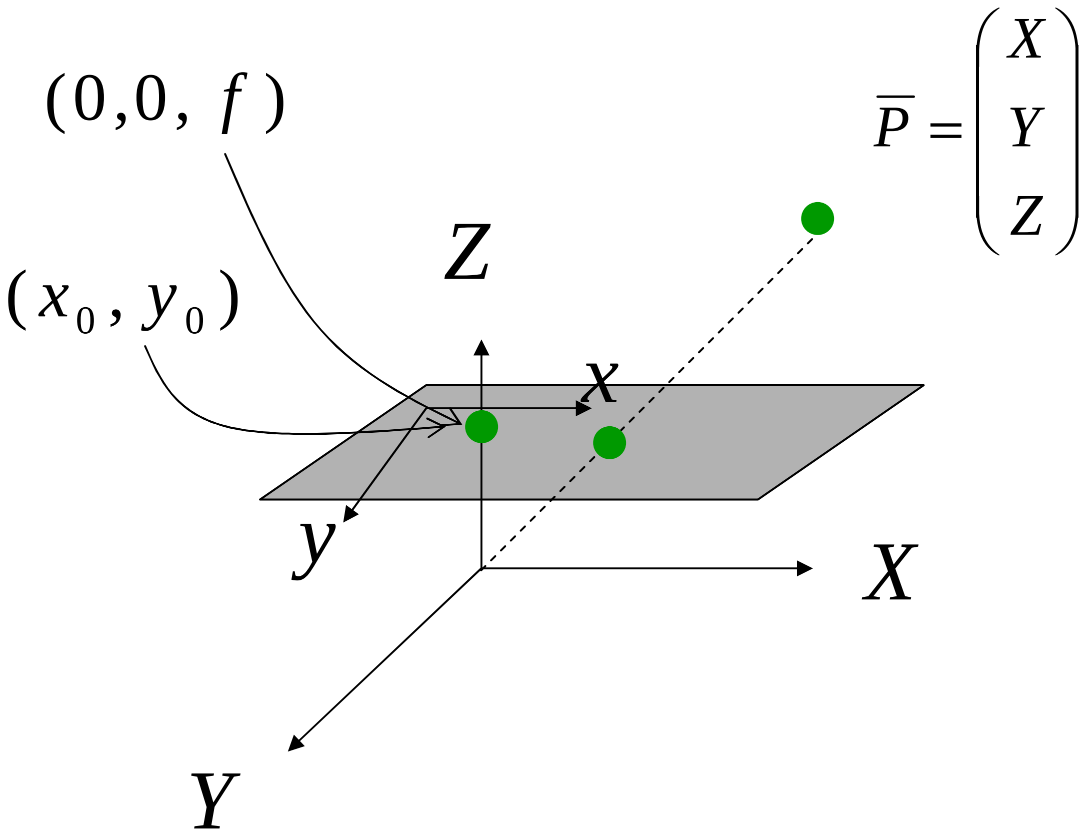




$$\overline{OP} = \overline{OO'} + U\vec{t} + V\vec{n} + W\vec{b}$$

$$R = \begin{bmatrix} \vec{t} & \vec{n} & \vec{b} \end{bmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} U \\ V \\ W \end{pmatrix} + T$$

# The Structure of a Projection Matrix




$$x = f\frac{X}{Z} + x_0$$

$$y = f\frac{Y}{Z} + y_0$$

# The Structure of a Projection Matrix

$$x = f\frac{X}{Z} + x_0$$

$$y = f\frac{Y}{Z} + y_0$$

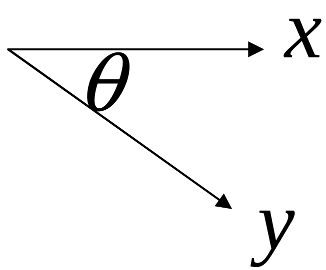
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = K(RP_w + T) = K[R;T] \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

$$p \cong M_{3\times 4} P$$

# The Structure of a Projection Matrix

Generally,

$$K = \begin{bmatrix} f_x & f_x \dfrac{\cos\theta}{\sin\theta} & x_0 \\ 0 & \dfrac{f_y}{\sin\theta} & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$




$(x_0, y_0)$ is called the "principle point"

$s = f_x \dfrac{\cos\theta}{\sin\theta}$ is called the "skew"

$\dfrac{f_y}{f_x}$ is "aspect ratio"

# The Camera Center

$$p \cong M_{3\times 4} P$$

$M$ has rank=3, thus $\exists c$ such that $Mc = 0$

$$K[R;T]c = 0 \Rightarrow [R;T]c = 0 \Rightarrow c = \begin{pmatrix} -R^T T \\ 1 \end{pmatrix}$$

Why is $c$ the camera center?

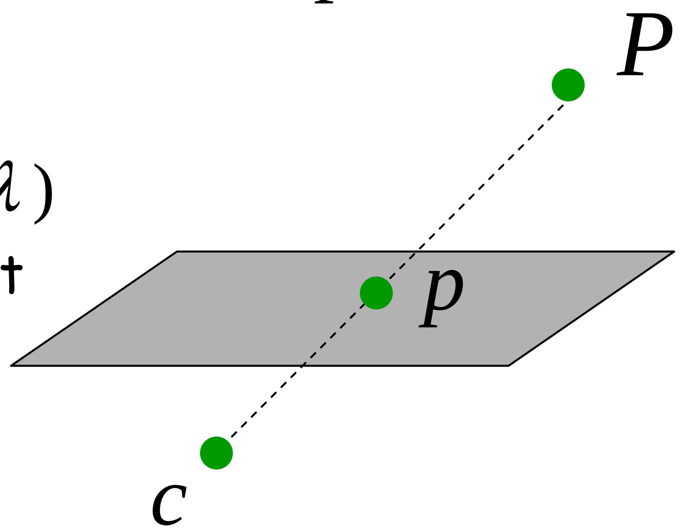
# The Camera Center

Why is $c$ the camera center?

Consider the "optical ray" $Q(\lambda) = \lambda P + (1 - \lambda)c$

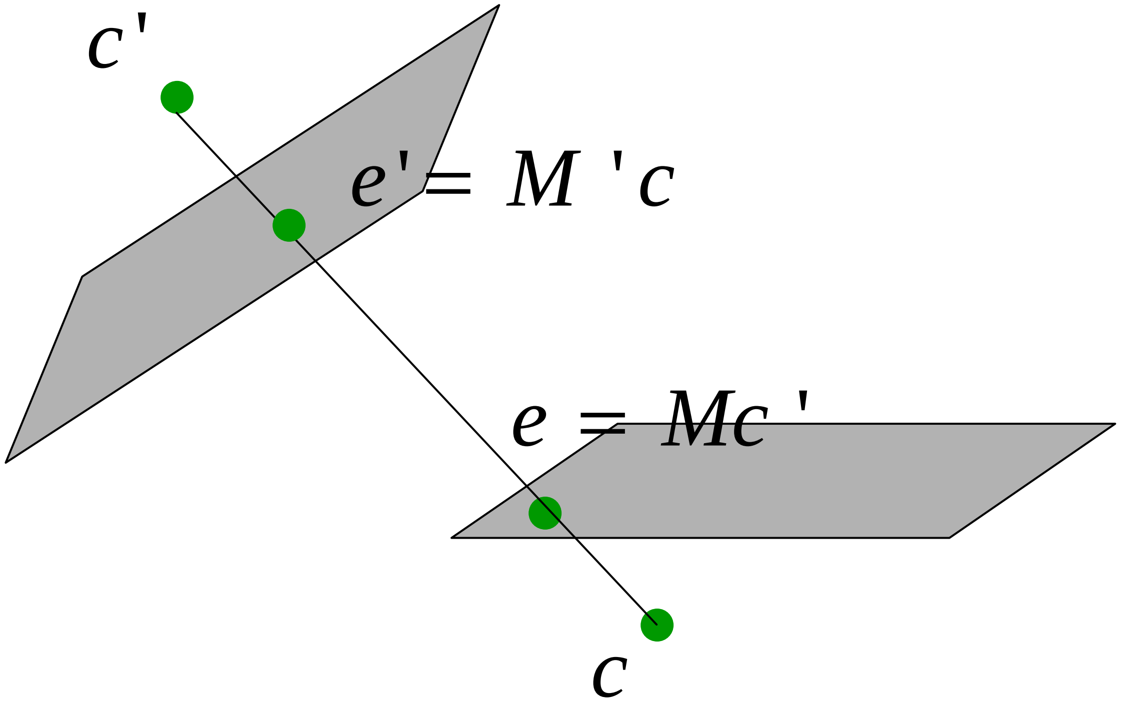
$$MQ(\lambda) = \lambda MP \cong MP \cong p$$

All points along the line $Q(\lambda)$ are mapped to the same point




$\Rightarrow$ $Q(\lambda)$ is a ray through the camera center

# The Epipolar Points



# Choice of Canonical Frame

$$p \cong MP = MWW^{-1}P$$

$$p' \cong M'P = M'WW^{-1}P$$

$W^{-1}P$ is the new world coordinate frame

We have 15 degrees of freedom (16 upto scale)

Choose W such that $MW = [I;0]$

# Choice of Canonical Frame

Let $$M = [M^*; m]$$

$$MW = [M^*; m]\begin{bmatrix} \overline{M}^{-1} - (\overline{M}m)n^T & -(1/\lambda)\overline{M}^{-1}m \\ n^T & 1/\lambda \end{bmatrix}$$

$$= \left[ I - mn^T + mn^T; \quad -(1/\lambda)m + (1/\lambda)m \right]$$

$$= \left[ I; \quad 0 \right]$$

We are left with 4 degrees of freedom (upto scale): $(n^T, \lambda)$

# Choice of Canonical Frame

$$p \cong [I;0]\overline{P}$$

$$p = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \overline{P} = \begin{pmatrix} p \\ \mu' \end{pmatrix}$$

$$p' \cong [H;e']P$$

$$p \cong [I;0]\begin{bmatrix} I & 0 \\ n^T & 1/\lambda \end{bmatrix}\begin{bmatrix} I & 0 \\ -\lambda n^T & \lambda \end{bmatrix}\overline{P}$$

$$\begin{bmatrix} I & 0 \\ -\lambda n^T & \lambda \end{bmatrix}\overline{P} = \begin{pmatrix} x \\ y \\ 1 \\ \mu \end{pmatrix} \equiv P$$

# Choice of Canonical Frame

$$p' \cong [H;e'] \begin{pmatrix} p \\ \mu' \end{pmatrix} = [H;e'] \begin{bmatrix} I & 0 \\ n^T & 1/\lambda \end{bmatrix} \begin{pmatrix} p \\ \mu \end{pmatrix}$$

$$[H;e'] \begin{bmatrix} I & 0 \\ n^T & 1/\lambda \end{bmatrix} = \left[ H + e'n^T; \quad (1/\lambda)e' \right]$$

$$\cong \left[ \lambda H + e'n^T; \quad e' \right]$$

$$\boxed{p' \cong \left[ \lambda H + e'n^T; \quad e' \right] \begin{pmatrix} p \\ \mu \end{pmatrix}}$$

where $(n^T, \lambda)$ are free variables

# Projection Matrices

Let $p_i^j$ be the image of point $P_i$ at frame number j

$$p_i^0 \cong [I;0]P_i$$

$$p_i^j \cong \left[\lambda H_j + e_j n^T; \quad e_j\right]P_i$$

where $(n^T, \lambda)$ are free variables

$p_i^j$ are known

# Family of Homography Matrices

$$p_i^j \cong \left[\lambda H_j + e_j n^T ; \quad e_j\right] P_i$$

$$H_\pi = \lambda H + e' n^T$$

$H_\pi$ Stands for the family of 2D projective transformations between two fixed images induced by a plane in space

The remainder of this class is about making the above statement intelligible

# Family of Homography Matrices

Recall,

$$p \cong K[R;T]\begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

3D->2D from Euclidean world frame to image

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R\begin{pmatrix} U \\ V \\ W \end{pmatrix} + T$$

world frame to first camera frame

Let K,K' be the internal parameters of camera 1,2 and choose canonical frame in which R=I and T=0 for first camera.

$$p \cong [K; \quad 0]P$$

$$p' \cong K'[R \quad t]P$$

$$P = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

# Family of Homography Matrices

$$p \cong [K; \quad 0]P$$

$$P = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$p' \cong K'[R \quad t]P$$

$$p = \frac{1}{Z} K \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Recall that 3rd row of K is $(0,0,1)$

$$\Rightarrow \quad p' \cong K'[R \quad t] \begin{pmatrix} ZK^{-1}p \\ 1 \end{pmatrix} = ZK'RK^{-1}p + K't$$

$$p' \cong K'RK^{-1}p + \frac{1}{Z}K't$$

# Family of Homography Matrices

$$p' \cong K'RK^{-1}p + \frac{1}{Z}K't$$

Assume $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ are on a planar surface $\bar{n}^T \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = d$

$\Rightarrow$ $\bar{n}^T (ZK^{-1}p) = d$

$$p' \cong K'RK^{-1}p + \frac{1}{Z}K't\left(\frac{Z\bar{n}^T K^{-1}p}{d}\right)$$

# Family of Homography Matrices

$$p' \cong K'RK^{-1}p + \frac{1}{Z}K't\left(\frac{Z\overline{n}^{T}K^{-1}p}{d}\right)$$

$$p' \cong (K'RK^{-1} + \frac{1}{d}K't\overline{n}^{T}K^{-1})\,p$$

Let $e' = K't,\quad n^{T} = \overline{n}^{T}K^{-1}$ and $H_{\infty} = K'RK^{-1}$

$$p' \cong (H_{\infty} + \frac{1}{d}e'n^{T})\,p$$

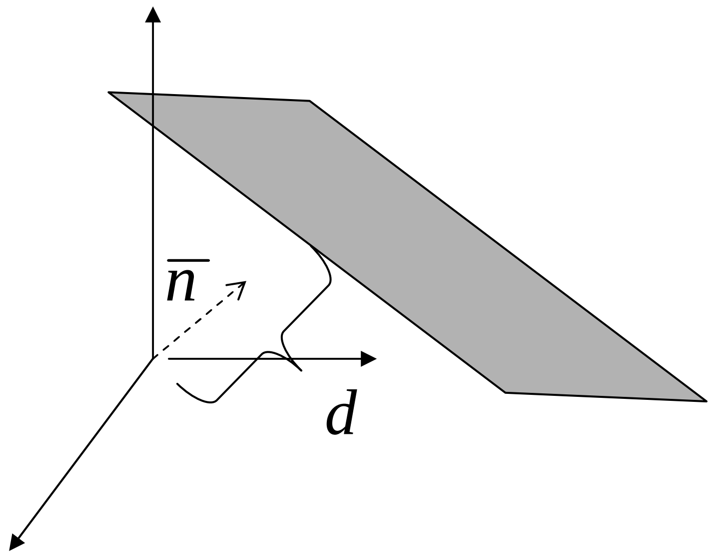
Image-to-image mapping $p \rightarrow p'$ where the matching pair $p, p'$ are induced by a planar surface.

# Family of Homography Matrices

$$H_{\pi} \cong H_{\infty} + \frac{1}{d} e' n^{T}$$

when $d \to \infty$

$$H_{\pi} \to H_{\infty} \cong K' R K^{-1}$$



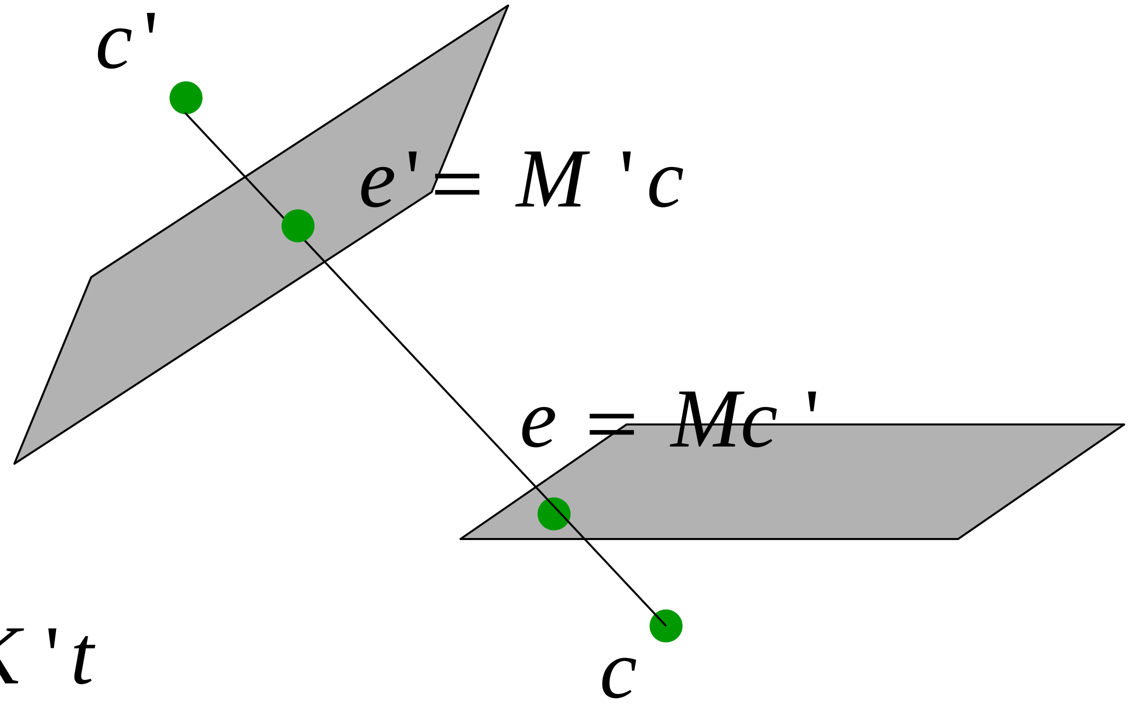
first camera frame

$\Rightarrow$ $K' R K^{-1}$ is the homography matrix induced by the plane at infinity

# Family of Homography Matrices

$$H_\pi \cong H_\infty + \frac{1}{d} e' n^T$$

$e' = K't$ is the epipole in the second image




$$M = [K; \quad 0]$$

$$M' = K'[R \quad t]$$

$$c^T = (0,0,0,1)$$

$$e' = M'c = K't$$

# Relationship Between two Homography Matrices

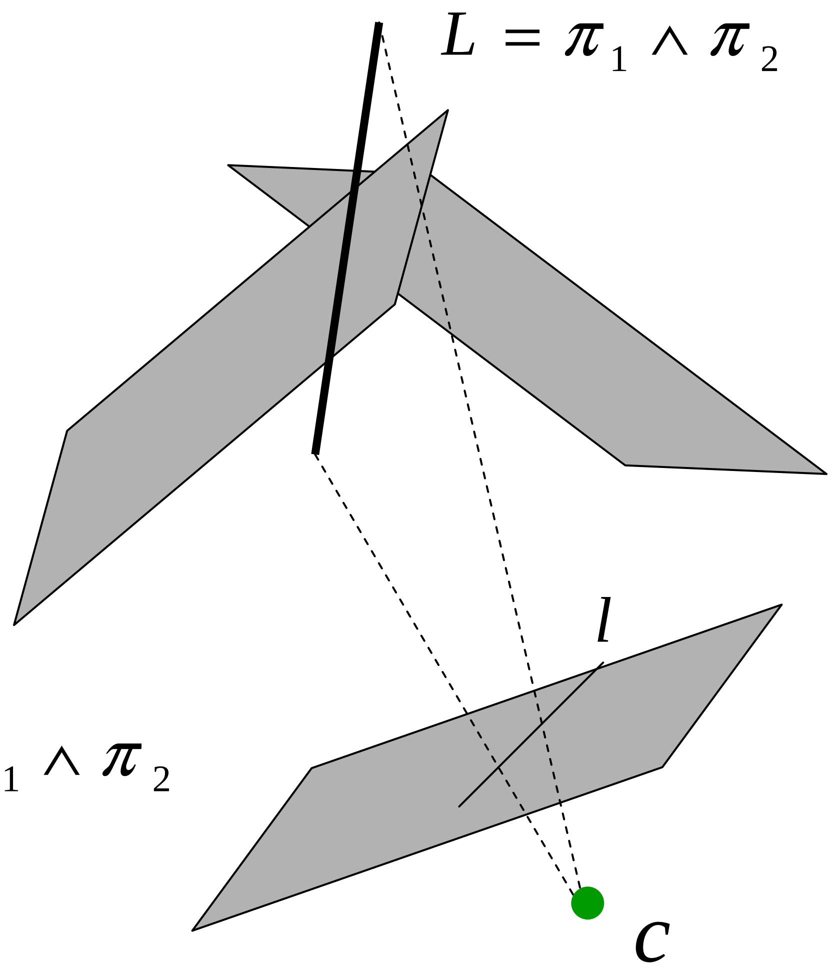
$$H_{\pi} \cong H_{\infty} + \frac{1}{d} e' n^T$$

$$\Rightarrow \quad H_{\pi_1} \cong \lambda H_{\pi_2} + e' l^T$$

$$H_{\pi_1} u \cong H_{\pi_2} u \qquad \forall u \in l$$

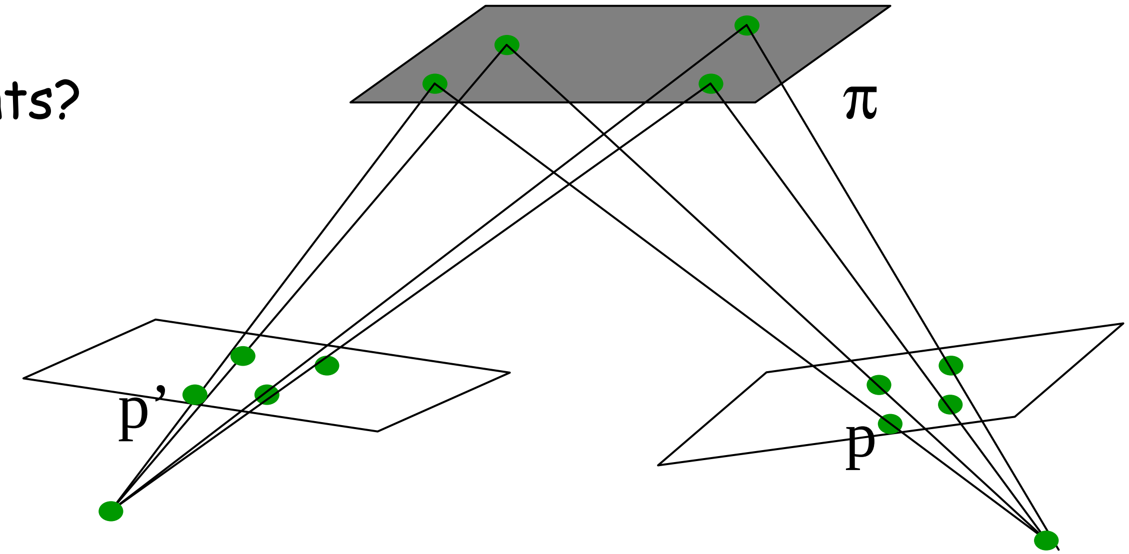
$$\Rightarrow \quad u^T l = 0$$

$\Rightarrow$ $l$ is the projection of $L = \pi_1 \wedge \pi_2$ onto the first image



# Estimating the Homography Matrix

How many matching points?




$$x' = \frac{h_1^T p}{h_3^T p}$$

$$p' \cong H_\pi p$$

4 points make a basis for the projective plane

$$y' = \frac{h_2^T p}{h_3^T p}$$

# Projective Geometry of the Plane

$ax + by + c = 0$ Equation of a line in the 2D plane

- The line is represented by the vector $l = (a, b, c)^T$
  and $p^T l = 0$ $\quad p = (x, y, 1)^T$

Correspondence between lines and vectors are not 1-1 because

$(\lambda a, \lambda b, \lambda c)^T$ represents the same line $p^T(\lambda l) = 0, \forall \lambda \neq 0$

The vector $(0,0,0)^T$ does represent any line.

- Two vectors differing by a scale factor are equivalent. This equivalence class is called homogenous vector. Any vector $(a, b, c)^T$ is a representation of the equivalence class.

# Projective Geometry of the Plane

A point $(x, y)$ lies on the line (coincident with) which is represented by $l = (a, b, c)^T$ iff $p^T l = 0$

But also $(\lambda p)^T l = 0$

$\Longrightarrow$ $(\lambda x, \lambda y, \lambda)^T, \forall \lambda \neq 0$ represents the point $(x, y)$

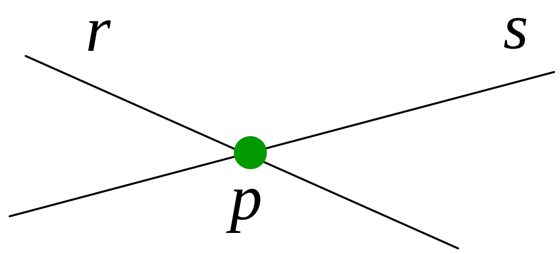
$\Longrightarrow$ $(x_1, x_2, x_3)^T$ represents the point $(\frac{x_1}{x_3}, \frac{x_2}{x_3})$

The vector $(0,0,0)^T$ does represent any point.

Points and lines are dual to each other (only in the 2D case!).

# Projective Geometry of the Plane

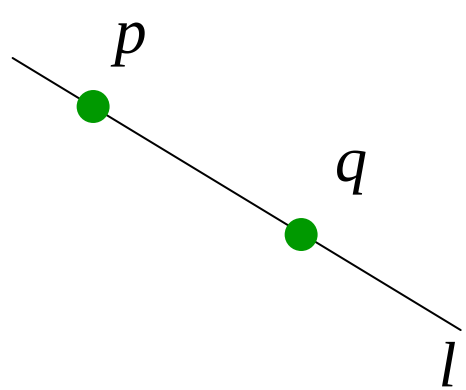
$$p \cong r \times s$$




$$p^T r = (r \times s)^T r = 0$$

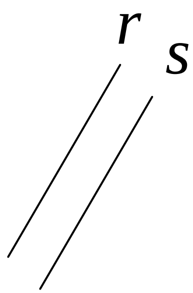
$$p^T s = (r \times s)^T s = 0$$

note: $(a \times b)^T c = \det(a, b, c)$




$$l \cong p \times q$$

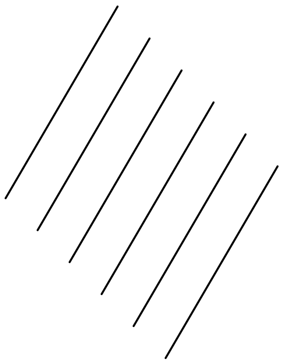
# Lines and Points at Infinity




Consider lines $r = (a, b, c)^T \quad s = (a, b, c')^T$

$$r \times s = \begin{pmatrix} bc' - cb \\ ac - ac' \\ 0 \end{pmatrix} = (c' - c)\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} \cong \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$$

which represents the point $(\frac{b}{0}, \frac{-a}{0})$ with infinitely large coordinates



$\begin{pmatrix} a \\ b \\ \lambda \end{pmatrix}$ All meet at the same point $\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$

# Lines and Points at Infinity

The points $(x_1, x_2, 0)^T, \forall x_1, x_2$ lie on a line

$$(x_1, x_2, 0)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

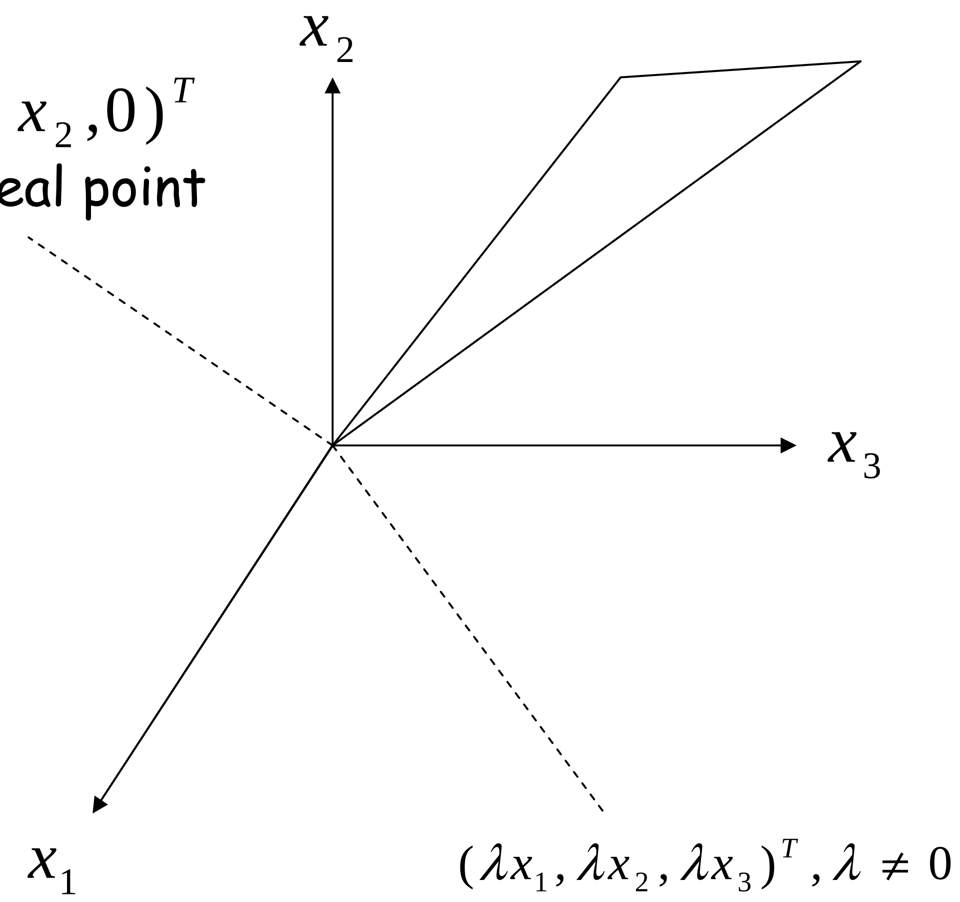
The line $l_\infty = (0,0,1)^T$ is called the line at infinity

The points $(x_1, x_2, 0)^T, \forall x_1, x_2$ are called ideal points.

A line $(a, b, \lambda)^T$ meets $l_\infty = (0,0,1)^T$ at $(b, -a, 0)^T$

(which is the direction of the line)

# A Model of the Projective Plane




$l_\infty = (0,0,1)^T$

is the plane $x_1 x_2$

Points are represented as lines (rays) through the origin

Lines are represented as planes through the origin

# A Model of the Projective Plane

$$\mathbf{P}^n = \{[x_1,\ldots, x_n] \neq [0,\ldots,0] : [x_1,\ldots, x_n] = [\lambda x_1,\ldots, \lambda x_n], \forall \lambda \neq 0\}$$

={lines through the origin in $R^{n+1}$ }

={1-dim subspaces of $R^{n+1}$ }

# Projective Transformations in $P^2$

The study of properties of the projective plane that are invariant under a group of transformations.

Projectivity: $h: P^2 \rightarrow P^2$

that maps lines to lines (i.e. preserves colinearity)

Any invertible 3x3 matrix is a Projectivity:

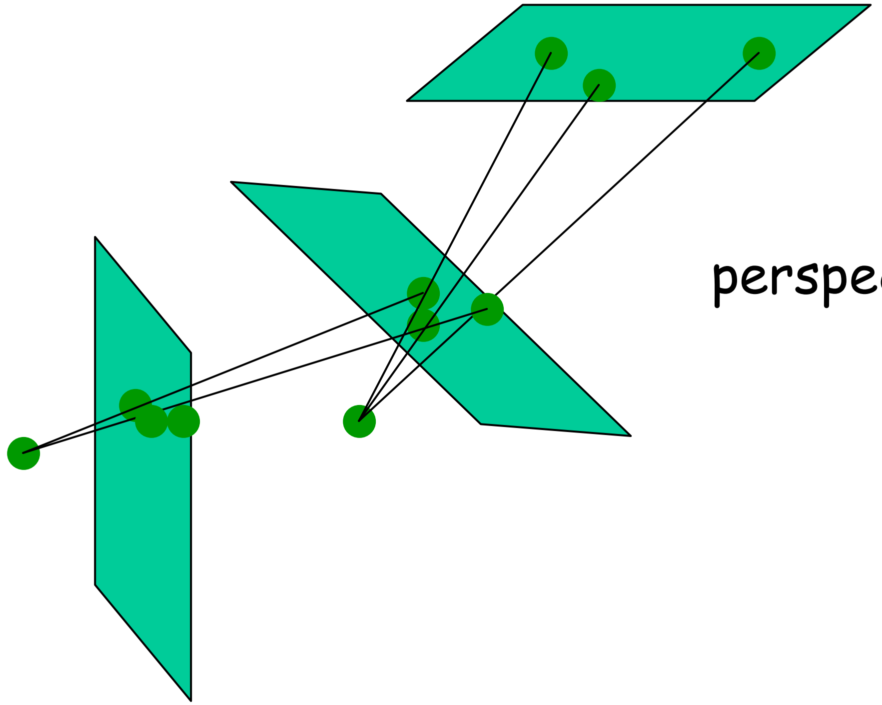
Let $p_1, p_2, p_3$ Colinear points, i.e.

$l^T p_i = 0 \Longrightarrow l^T H^{-1} H p_i = 0$

$\Longrightarrow$ the points $Hp_i$ lie on the line $H^{-T} l$

$H$ is called homography, colineation $H^{-T}$ is the dual.

# Projective Transformations in $P^2$



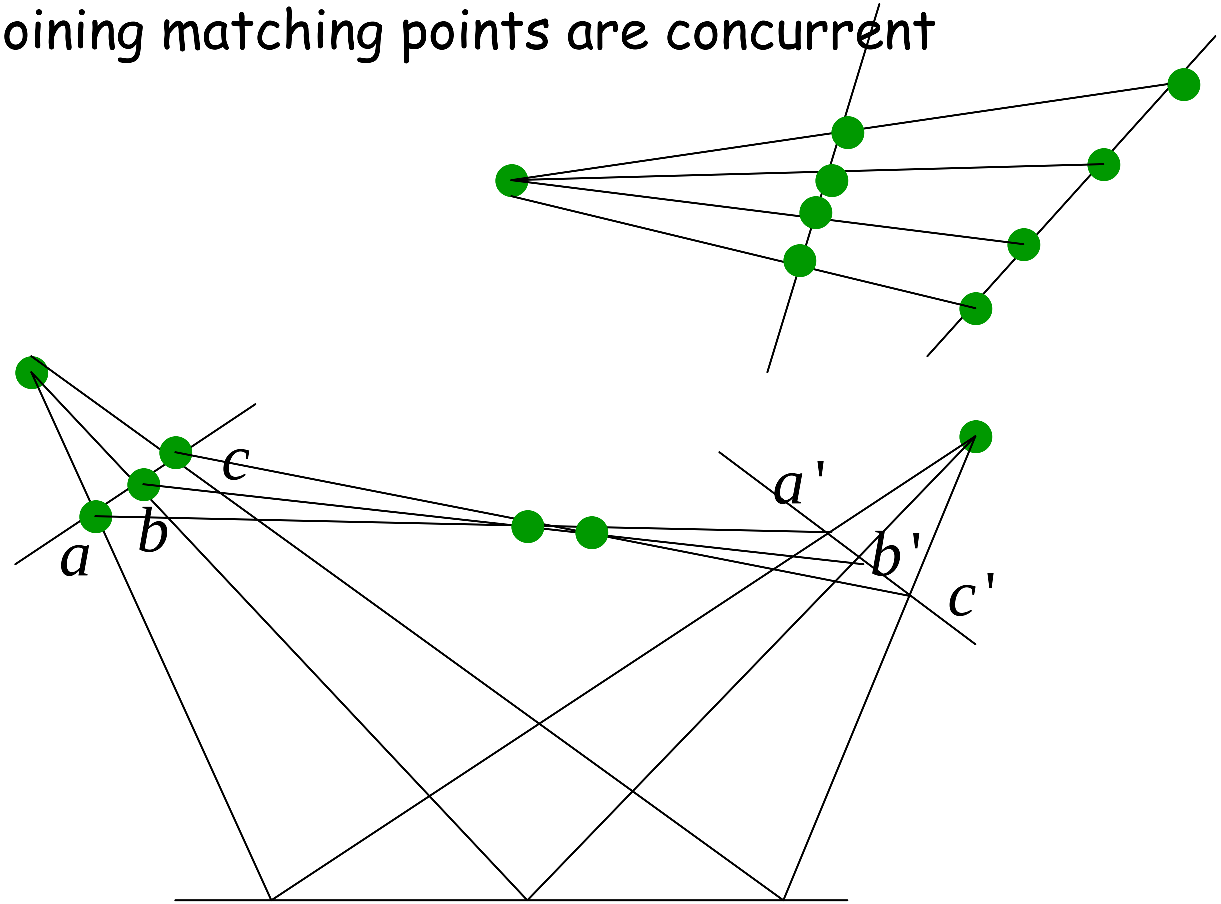

A composition of perspectivities from a plane $\pi$ to other planes and back to $\pi$ is a projectivity.

Every projectivity can be represented in this way.

# Projective Transformations in $P^2$

Example, a prespectivity in 1D:

Lines adjoining matching points are concurrent




Lines adjoining matching points (a,a'),(b,b'),(c,c') are not concurrent

# Projective Transformations in $P^2$

$l_\infty = (0,0,1)^T$ is not invariant under H:

Points on $l_\infty$ are $(x_1, x_2, 0)^T$

$$H\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 h_1 + x_2 h_2 = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

$x'_3$ is not necessarily 0

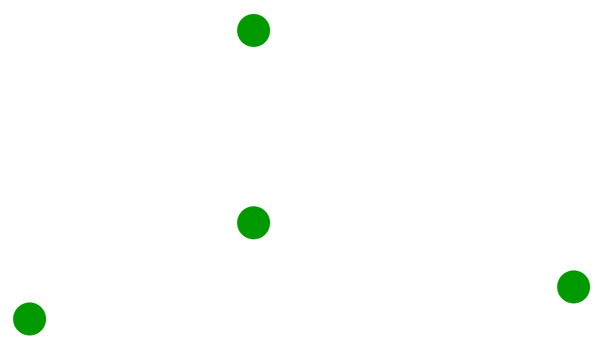
⇒ Parallel lines do not remain parallel !

⇒ $l_\infty$ is mapped to $H^{-T} l_\infty$

# Projective Basis

A Simplex in $R^{n+1}$ is a set of n+2 points such that no subset
Of n+1 of them lie on a hyperplane (linearly dependent).

In $P^2$ a Simplex is 4 points



Theorem: there is a unique colineation between any two Simplexes

# Why do we need 4 points

Invariants are measurements that remain fixed under colineations

# of independent invariants = # d.o.f of configuration - # d.o.f of trans.

Ex: 1D case $p' \cong H_{2x2}\, p$ $\Longrightarrow$ H has 3 d.o.f

A point in 1D is represented by 1 parameter.

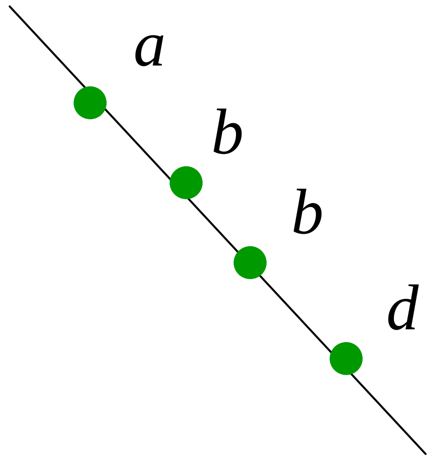
4 points we have: 4-3=1 invariant (cross ratio)

2D case: H has 8 d.o.f, a point has 2 d.o.f thus 5 points induce 2 invariants

# Why do we need 4 points

The cross-ratio of 4 points:

$$\alpha = \frac{\frac{\overline{ab}}{\overline{ac}}}{\frac{\overline{cd}}{\overline{bd}}}$$




24 permutations of the 4 points forming 6 groups:

$$\alpha, \frac{1}{\alpha}, 1-\alpha, \frac{\alpha-1}{\alpha}, \frac{\alpha}{\alpha-1}, \frac{1}{1-\alpha}$$

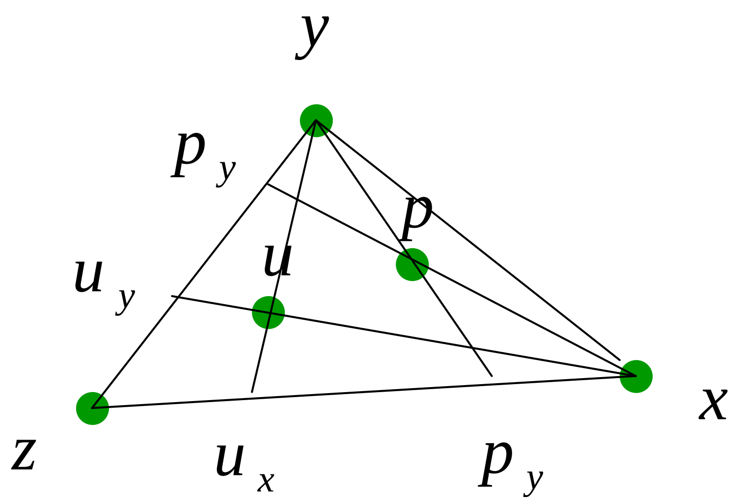
# Why do we need 4 points

5 points gives us 10 d.o.f, thus 10-8=2 invariants which represent 2D

$x, y, z, u$ are the 4 basis points (simplex)

$$\alpha =< z, u_x, p_x, x >$$

$$\beta =< z, u_y, p_y, y >$$




$\Rightarrow$ $p_x, p_y$ are determined uniquely by $\alpha, \beta$

Point of intersection is preserved under projectivity (exercise)

$\Rightarrow$ $p_x, p_y$ uniquely determined $p$