Multi-linear Systems for 3D-from-2D Interpretation

Lecture 3

Dynamic $P^n \rightarrow P^n$ Tensors

Amnon Shashua

Hebrew University of Jerusalem
Israel

Material We Will Cover Today

- 2D-2D, 3D-D mapping of a “dynamic” point configuration

- Homography Tensors and their properties

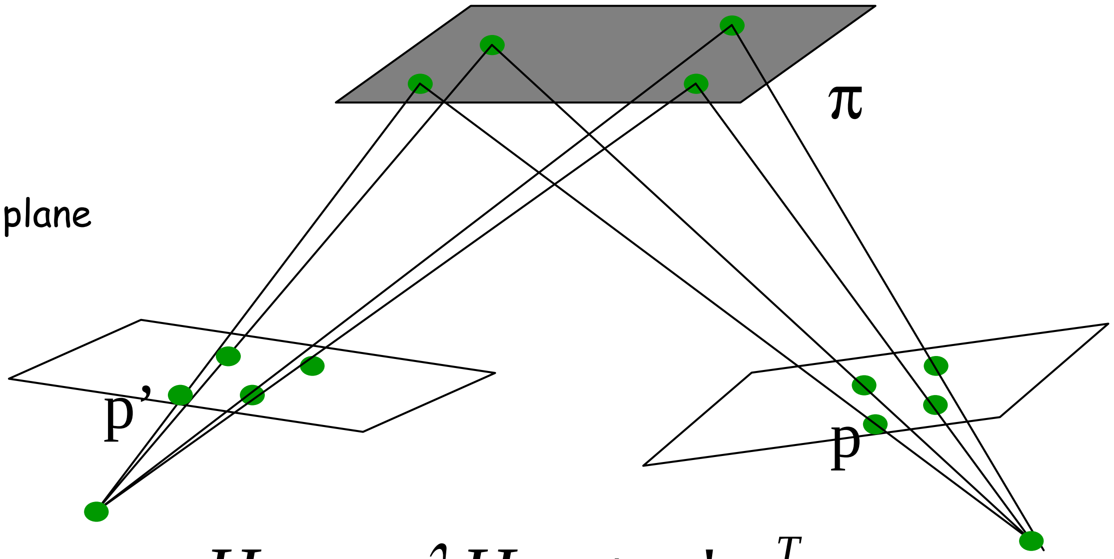
- Some generalizations to higher dimensions

Shashua,Wolf ECCV’00	(2D-2D,best paper award)
Wolf,Shashua,Wexler ICPR’00	(3D-3D)
Shashua,Meshulam,Wolf,Levin,Kalai ‘02	(generalizations to high dimensions)
Shashua,Wolf ‘04	(extended version)

Homography Matrix

$$p' \cong H_{\pi} p$$

4 points make a basis for the projective plane



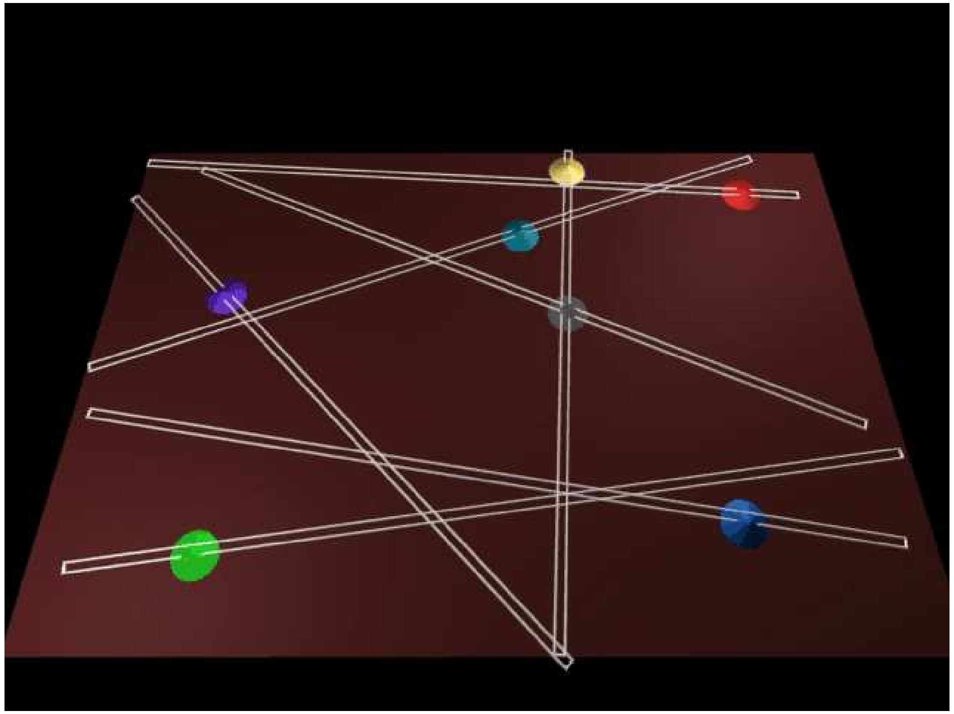
$$H_{\pi} = \lambda H_{\infty} + e' n^{T}$$

H_{π} Stands for the family of 2D projective transformations between two fixed images induced by a plane in space

Does is make "sense" to introduce a 3rd view?

$P^2 \otimes P^2 \to P^2$ Mapping of the dynamic projective plane onto itself

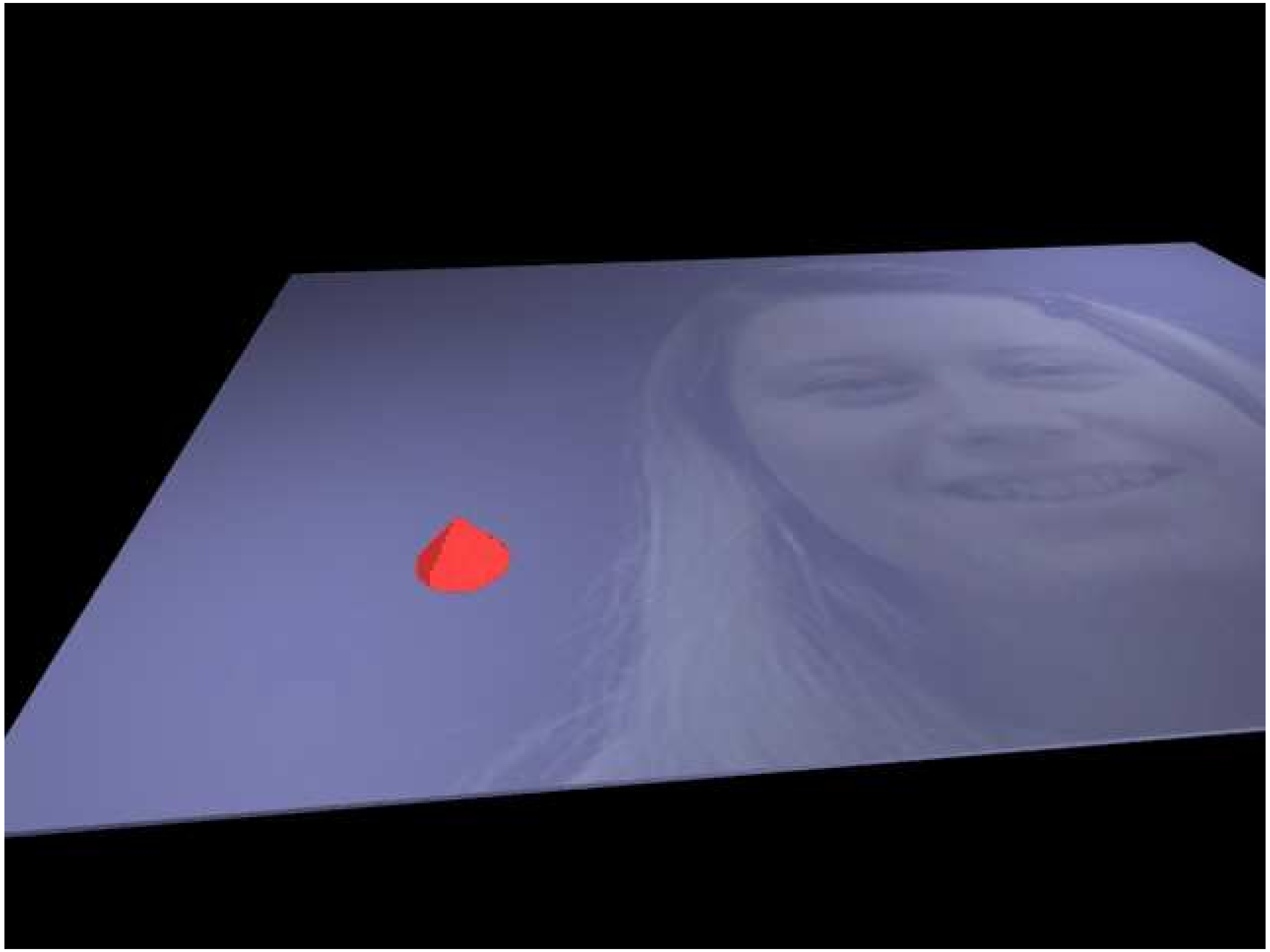
(movie)



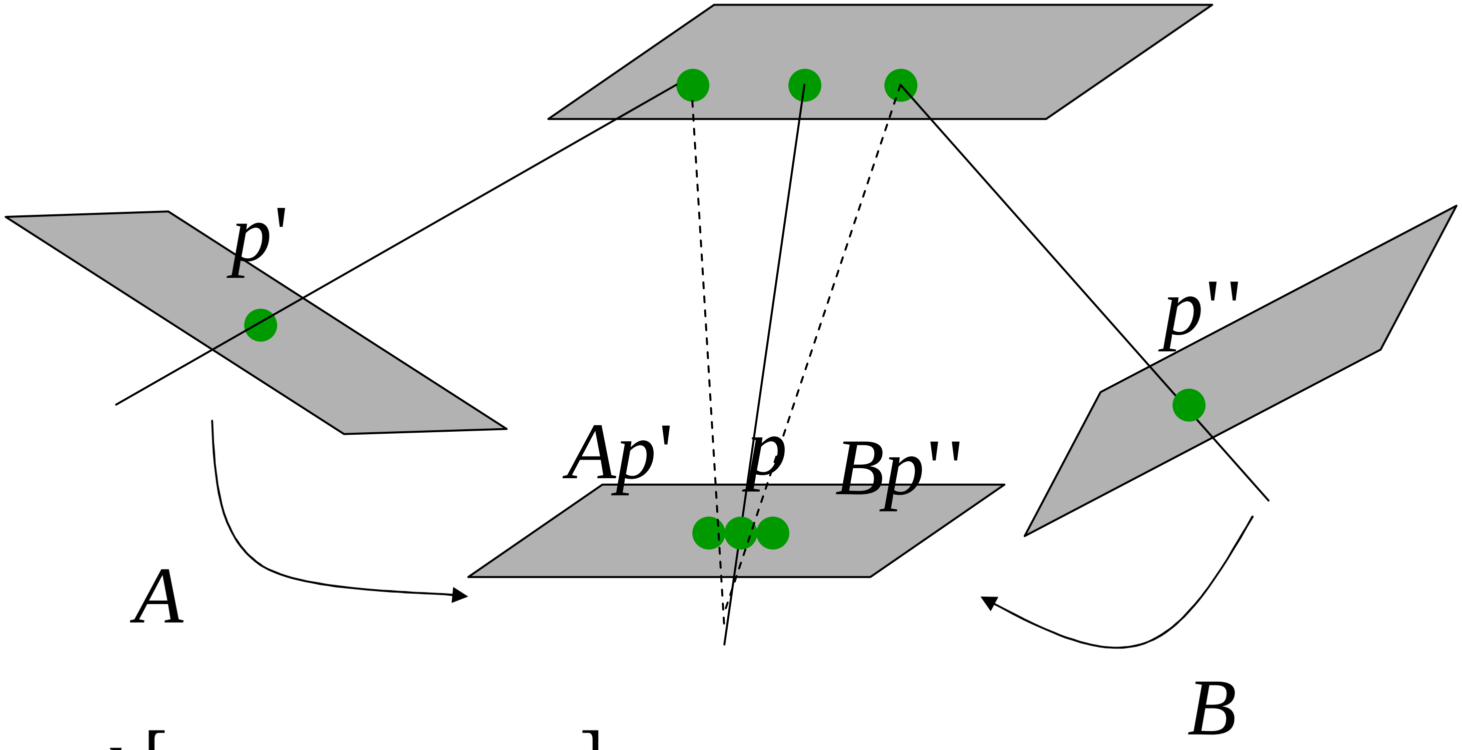
Points are moving along straight-line trajectories
while camera changes position

3 snapshots of a linearly moving point

(movie)



A,B are unknown



$$rank\begin{bmatrix} p & Ap' & Bp'' \end{bmatrix} = 2$$

$\Longrightarrow$ $$p^T (Ap' \times Bp'') = 0$$

Multilinear relation between p,p',p'' and A,B

$$p^T(Ap' \times Bp'')=0$$

Let index i run over view 1, index j over view 2 and index k over view 3

$$p^i \varepsilon_{inu}(a_j{}^n p'^j)(b_k{}^u p''^k)=0$$

The position of symbols does not matter (only the indices)

$$p^i p'^j \, p''^k (\varepsilon_{inu} a_j{}^n b_k{}^u)=0$$

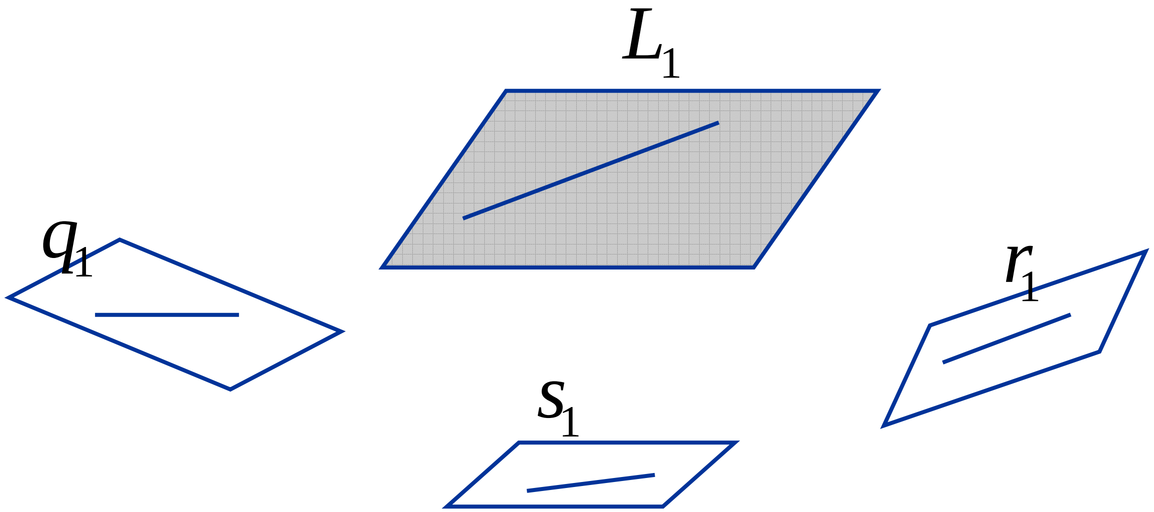
$$\Rightarrow \quad p^i p'^j \, p''^k H_{ijk}=0$$

Estimation of H_{ijk}

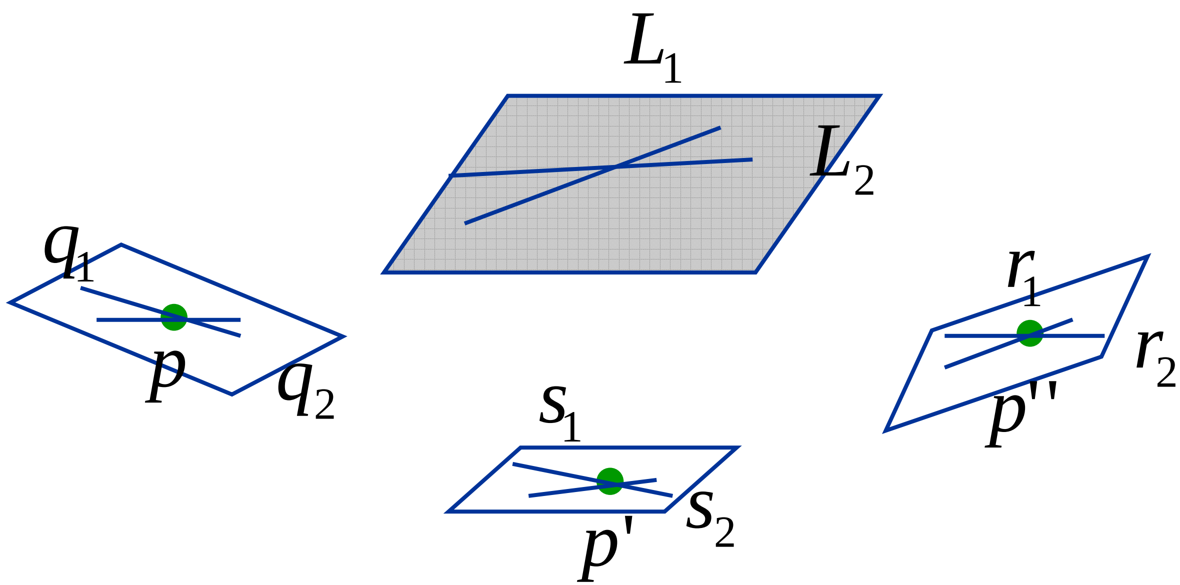
- 26 matching triplets p,p’,p’’ arising from dynamic points provide A unique solution for the dual Htensor (each triplet provides one linear constraint).

- The 26 points must lie on *at least* 4 lines (in general position), where no more than 8 points on the first line, no more than 7 points on the second line, 6 on the third, and 5 on the fourth.

(sketch of proof to follow)



Consider a line L_1 and its projections q_1, s_1, r_1. Since each line is determined by two points we can have at most $2^3 = 8$ linearly independent constraints $p^i p'^j p''^k H_{ijk} = 0$ where the points p, p', p'' are coincident with the lines q_1, s_1, r_1 respectively.



Consider a second line L_2 and its projections q_2, s_2, r_2. Let p, p', p'' be a triplet of points of the projection of $L_1 \cap L_2$

Among the 8 possible independent choices of choosing three points From the three pairs of points (defining the lines q_2, s_2, r_2) the choice is already covered by the span of the 8 constraints p, p', p'' ated with L_1.

We are left with only 7 linearly independent constraints induced by L_2

Labeled Stationary Points

$$rank[p \quad Ap' \quad Bp''] = 1$$

$$p^i p'^j H_{ijk} = 0 \qquad p^i p''^k H_{ijk} = 0 \qquad p'^j p''^k H_{ijk} = 0$$

➡ 9 linear constraints on H

$$p^i p'^j e^k H_{ijk} = 0, \forall e$$
$$p^i p''^k e^j H_{ijk} = 0, \forall e$$
$$p'^j p''^k e^i H_{ijk} = 0, \forall e$$

But $p^i p'^j p''^k H_{ijk} = 0$ Appears 3 times!

➡ 7 linearly independent constraints on H

➡ 4 (labeled) static points are sufficient for solving for H

Unlabeled Stationary Points

What if all the measurements arise from stationary points without prior knowledge that they are stationary? (unlabeled stationary)

$$p^i p'^j p''^k H_{ijk} = 0 \quad \Longrightarrow \quad G^{ijk} H_{ijk} = 0$$

It is sufficient to consider A=B=I $\Longrightarrow$ $G^{ijk} = p^i p^j p^k$

G^{ijk} is a symmetric tensor, i.e. contains only 10 different groups

111,222,333,112,113,221,223,331,332,123

up to permutations. $rank(v_1 \otimes v_2 \otimes v_3 \mid rankspan(v_1, v_2, v_3) = 1) = \binom{n+3-1}{3}$

$\Longrightarrow$ One needs at least 16 dynamic points in an unlabeled set

Mixed Labeled and Unlabeled Static Points

Let $0 \le x \le 4$ be the number of labeled stationary points. To obtain a unique linear solution for the homography tensor, the minimal number of matching triplets associated with moving points must be $16-4x$ and at most $10-3x$ can come from unlabeled stationary points.

$p^i p'^j p''^k H_{ijk} = 0 \implies G^{ijk} H_{ijk} = 0$ It is sufficient to consider A=B=I $\implies G^{ijk} = p^i p^j p^k$

Consider the case $x=1$, i.e., one of the matching triplets contributed 9 constraints of rank 7:

$$p^i p'^j e_1^k H_{ijk} = 0 \qquad p^i e_1^j p''^k H_{ijk} = 0 \qquad e_1^i p'^j p''^k H_{ijk} = 0$$

$$p^i p'^j e_2^k H_{ijk} = 0 \qquad p^i e_2^j p''^k H_{ijk} = 0 \qquad e_2^i p'^j p''^k H_{ijk} = 0$$

$$p^i p'^j e_3^k H_{ijk} = 0 \qquad p^i e_3^j p''^k H_{ijk} = 0 \qquad e_3^i p'^j p''^k H_{ijk} = 0$$

Where e_1, e_2, e_3 are the standard basis $(1,0,0),(0,1,0),(0,0,1)$

Mixed Labeled and Unlabeled Static Points

Add the three constraints of the first row: $E^{ijk} = p^i p^j e_1^k + p^i e_1^j p^k + e_1^k p^j p^k$

Then, E^{ijk} is a symmetric tensor and is therefore spanned by the 10-dim subspace of the unlabeled stationary points.

Consider the case $x = 1$, i.e., one of the matching triplets contributed 9 constraints of rank 7:

$p^i p'^j e_1^k H_{ijk} = 0$	$p^i e_1^j p''^k H_{ijk} = 0$	$e_1^i p'^j p''^k H_{ijk} = 0$
$p^i p'^j e_2^k H_{ijk} = 0$	$p^i e_2^j p''^k H_{ijk} = 0$	$e_2^i p'^j p''^k H_{ijk} = 0$
$p^i p'^j e_3^k H_{ijk} = 0$	$p^i e_3^j p''^k H_{ijk} = 0$	$e_3^i p'^j p''^k H_{ijk} = 0$

Where e_1, e_2, e_3 are the standard basis $(1,0,0), (0,1,0), (0,0,1)$

Mixed Labeled and Unlabeled Static Points

Add the three constraints of the first row: $E^{ijk} = p^i p^j e_1^k + p^i e_1^j p^k + e_1^k p^j p^k$

Then, E^{ijk} is a symmetric tensor and is therefore spanned by the 10-dim subspace of the unlabeled stationary points.

Likewise, the constraint tensors resulting from summing the second and third rows Are symmetric. As a result 3 out of the 7 constraints contributed by a labeled stationary Points are already accounted for by the space of unlabeled stationary points.

Consider the case $x=1$, i.e., one of the matching triplets contributed 9 constraints of rank 7:

$p^i p'^j e_1^k H_{ijk} = 0$	$p^i e_1^j p''^k H_{ijk} = 0$	$e_1^i p'^j p''^k H_{ijk} = 0$
$p^i p'^j e_2^k H_{ijk} = 0$	$p^i e_2^j p''^k H_{ijk} = 0$	$e_2^i p'^j p''^k H_{ijk} = 0$
$p^i p'^j e_3^k H_{ijk} = 0$	$p^i e_3^j p''^k H_{ijk} = 0$	$e_3^i p'^j p''^k H_{ijk} = 0$

Where e_1, e_2, e_3 are the standard basis $(1,0,0), (0,1,0), (0,0,1)$

Mixed Labeled and Unlabeled Static Points

Add the three constraints of the first row: $E^{ijk} = p^i p^j e_1^k + p^i e_1^j p^k + e_1^k p^j p^k$

Then, E^{ijk} is a symmetric tensor and is therefore spanned by the 10-dim subspace of the unlabeled stationary points.

Likewise, the constraint tensors resulting from summing the second and third rows Are symmetric. As a result 3 out of the 7 constraints contributed by a labeled stationary Points are already accounted for by the space of unlabeled stationary points.

Consider the case $x = 1$, i.e., one of the matching triplets contributed 9 constraints of rank 7:

$p^i p'^j e_1^k H_{ijk} = 0$	$p^i e_1^j p''^k H_{ijk} = 0$	$e_1^i p'^j p''^k H_{ijk} = 0$
$p^i p'^j e_2^k H_{ijk} = 0$	$p^i e_2^j p''^k H_{ijk} = 0$	$e_2^i p'^j p''^k H_{ijk} = 0$
$p^i p'^j e_3^k H_{ijk} = 0$	$p^i e_3^j p''^k H_{ijk} = 0$	$e_3^i p'^j p''^k H_{ijk} = 0$

Where e_1, e_2, e_3 are the standard basis $(1,0,0), (0,1,0), (0,0,1)$

Mixed Labeled and Unlabeled Static Points

- 3 of the 7 constraints provided by a labeled static live in the 10'th dimensional subspace of unlabeled static points.

⇓

- If we have $0 \leq x \leq 4$ labeled static points, then we need

$16 - 4x$ dynamic points

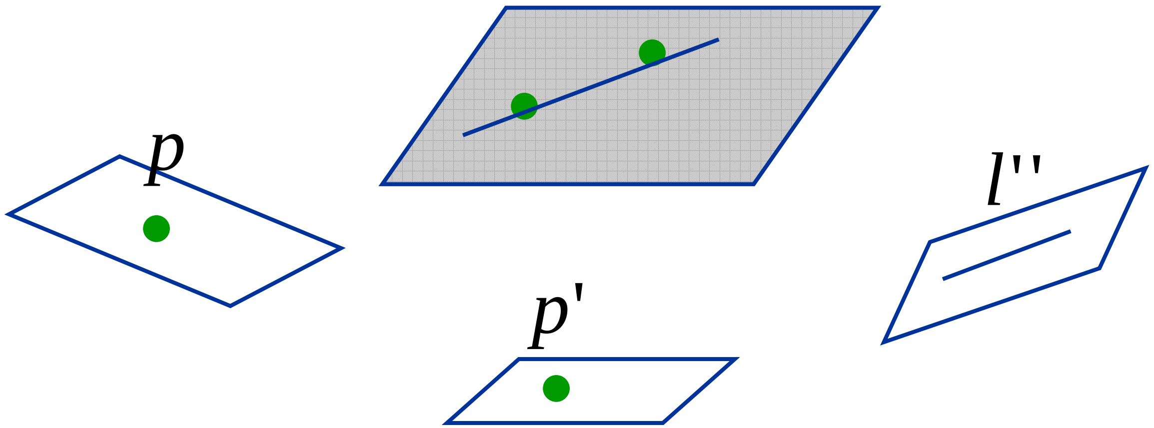
Partly segmented scene

Known static	unknown	moving required
0	26	16
1	19	12
2	12	8
3	5	4
4	0	0

H_{ijk} as a mapping and its slices

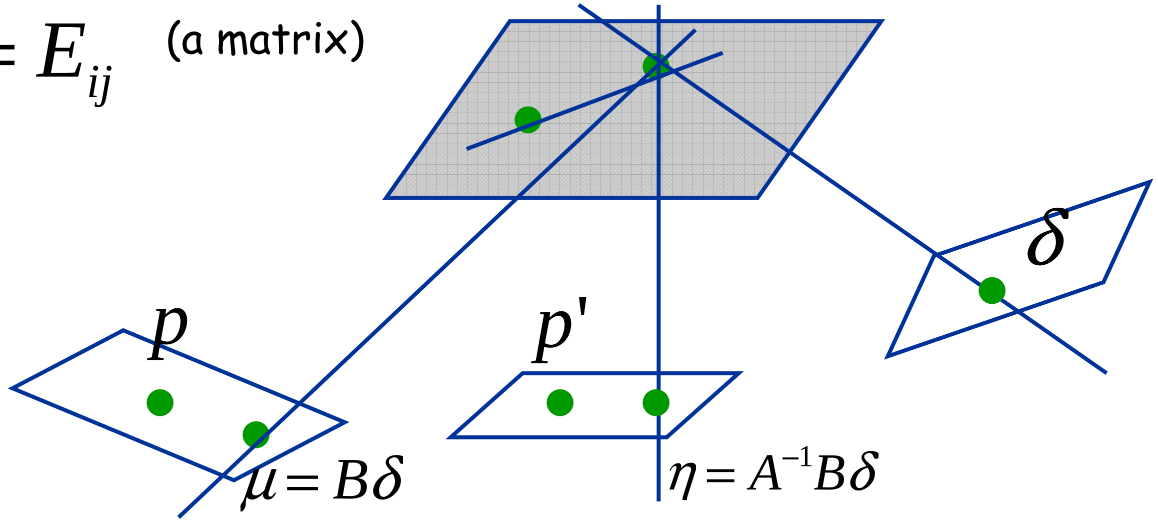
Double contraction: point-point to line mapping

$$p^i p'^j H_{ijk} = l''_k$$



From index structure l'' must be a line. Since $p^i p'^j p''^k H_{ijk} = 0$ for every point along the straight line trajectory determined by p,p' then the line l'' must the projection of that trajectory.

Single contraction: $\delta^k H_{ijk} = E_{ij}$ (a matrix)



$$H_{ijk} = \varepsilon_{inu} a_j{}^n b_k{}^u$$

$$\delta^k H_{ijk} = \varepsilon_{inu} a_j{}^n (b_k{}^u \delta^k) = [B\delta]_\times A$$

$\Longrightarrow$

$$E = [\mu]_x A \qquad \mu = B\delta$$

$$E\eta = [\mu]_x A\eta \cong [\mu]_x \mu = 0$$

$$E^T \mu = -A^T [\mu]_x \mu = 0$$

$$p^T E p' = 0$$

For all pairs p,p' on matching lines through the fixed points μ and η

The single contraction is the key for recovering A,B:

$$E = [\mu]_x A$$

$$E^T A = -A^T [\mu]_x A$$

$$A^T E = A^T [\mu]_x A$$

$$\Longrightarrow \quad A^T E + E^T A = 0$$

Given E we obtain 6 linear equations to solve for A

With 2 slices,

$$\begin{pmatrix}1\\0\\0\end{pmatrix}^k H_{ijk} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix}^k H_{ijk}$$

one can solve for A

To summarize, each of the contractions:

$$\delta^k H_{ijk}$$

$$\delta^j H_{ijk}$$

$$\delta^i H_{ijk}$$

Represents a point-to-line (correlation) mapping between views (1,2),(1,3) and (2,3) respectively. By setting δ to be $(1,0,0),(0,1,0),(0,0,1)$ we obtain three different slicings of the tensor.

Let G_1, G_2, G_3 be the slices of $\delta^i H_{ijk}$

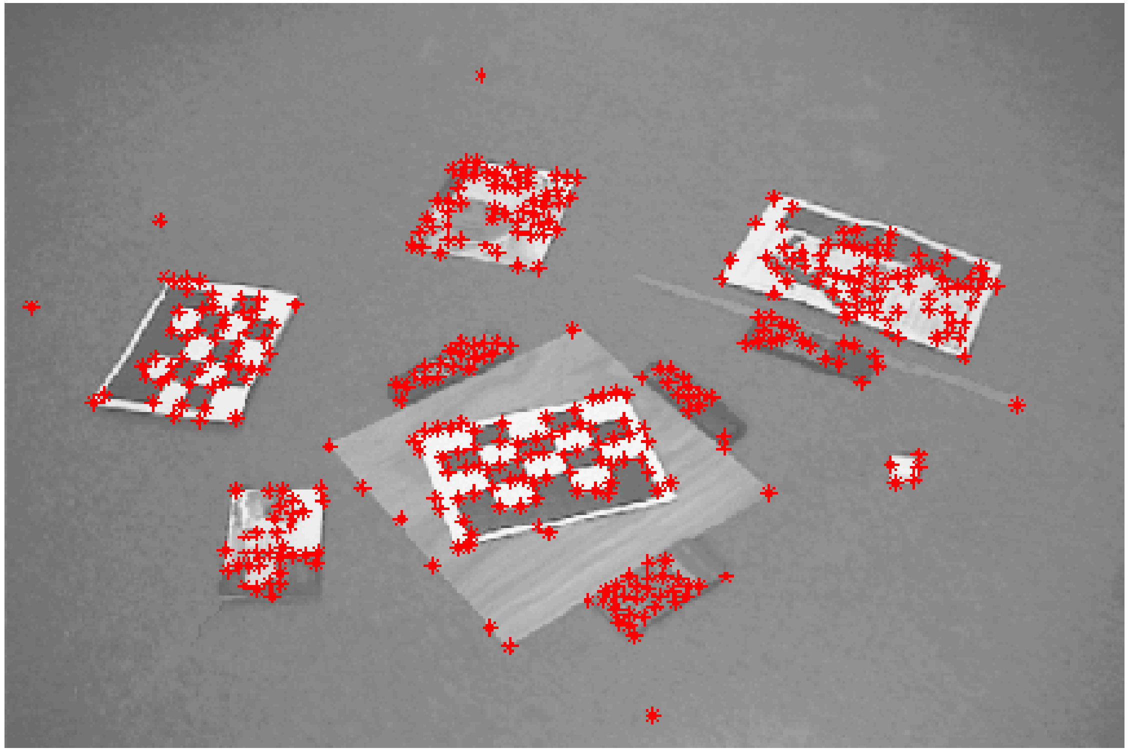
W_1, W_2, W_3 be the slices of $\delta^j H_{ijk}$

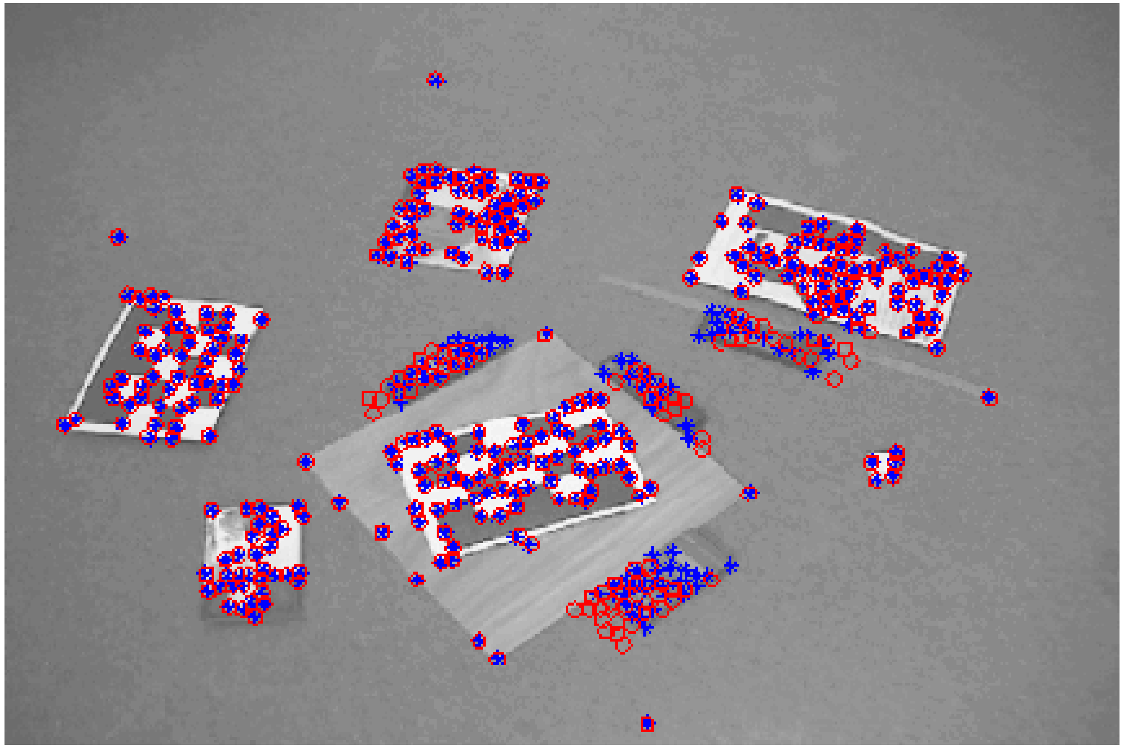
E_1, E_2, E_3 be the slices of $\delta^k H_{ijk}$

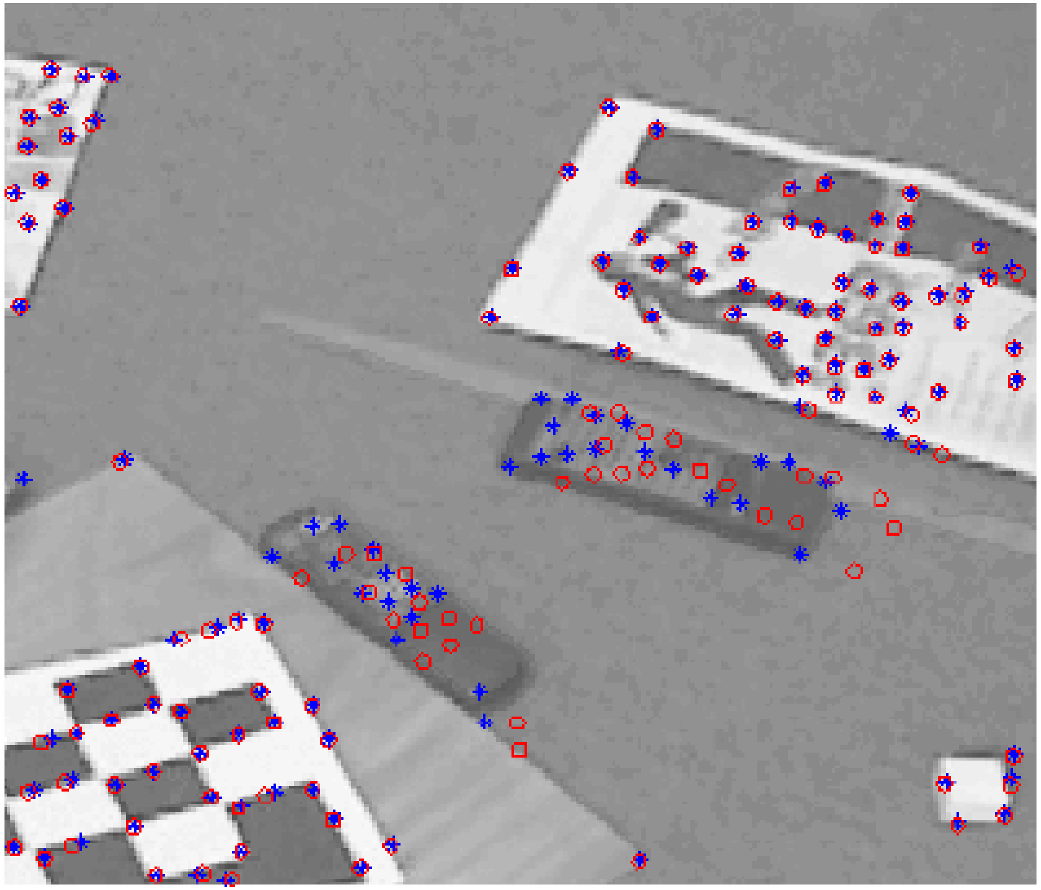
Then:

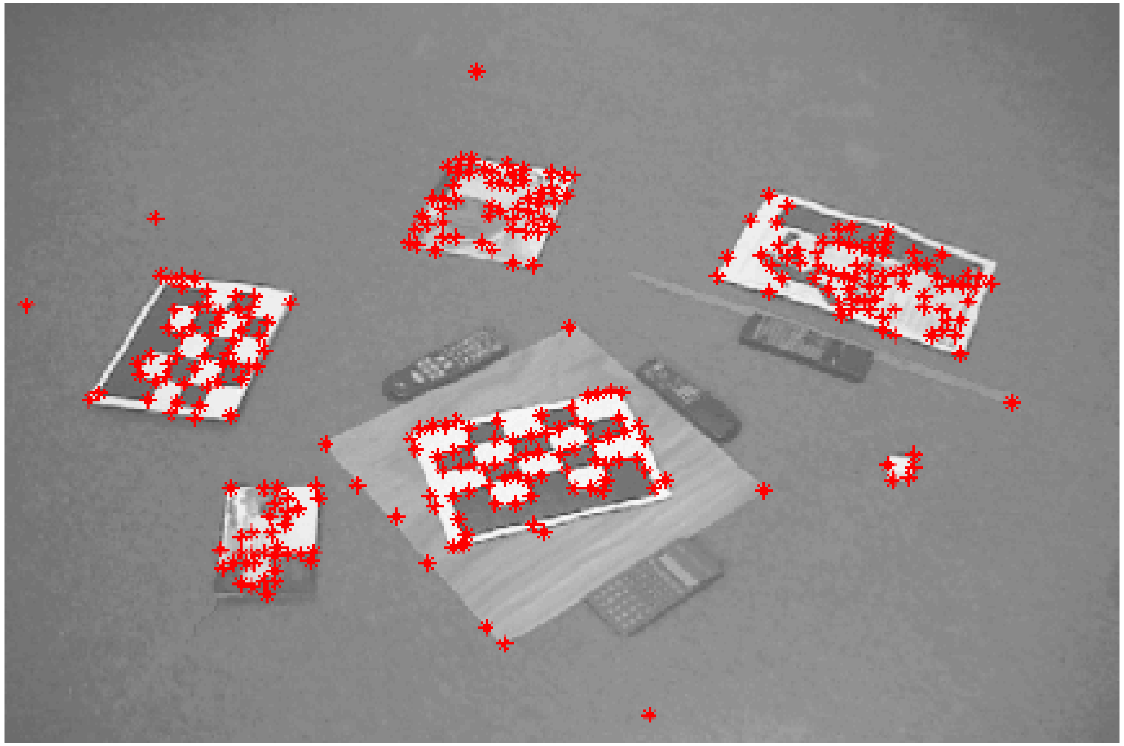
$$\begin{aligned} BW_i^\top + W_i B^\top &= 0, \\ CG_i^\top + G_i C^\top &= 0, \\ AE_i^\top + E_i A^\top &= 0 \end{aligned} \qquad C = A^{-1}B$$

(movie)



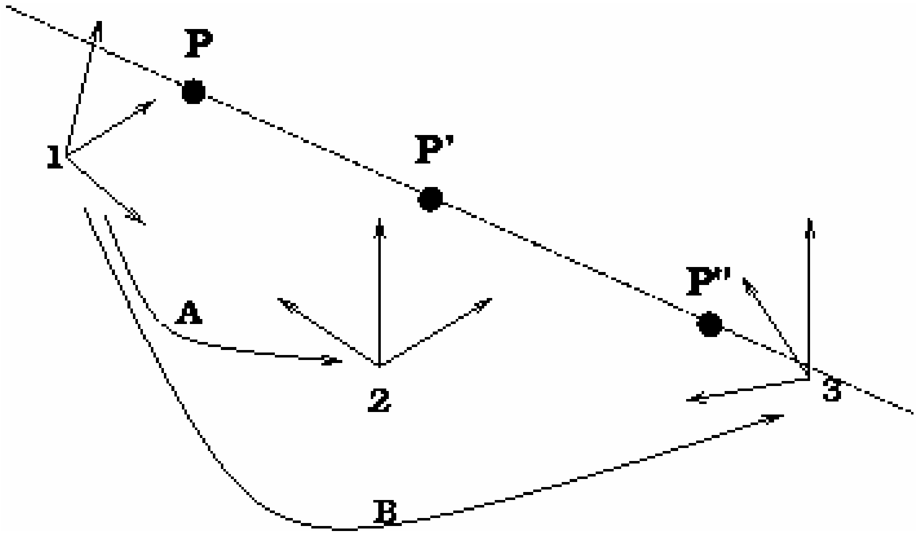




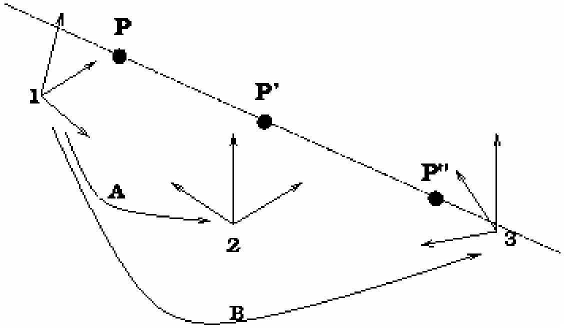


Homography Tensors of P^3

QuickTime™ and a
decompressor
are needed to see this picture.



Homography Tensors of P^3



Basic constraint: $$rank\begin{pmatrix} | & | & | \\ P & AP' & BP'' \\ | & | & | \end{pmatrix} = 2$$

For every vector V: $$\det\begin{pmatrix} | & | & | & | \\ P & AP' & BP'' & V \\ | & | & | & | \end{pmatrix} = 0$$

$$A = \begin{bmatrix} X_1 & Y_1 & Z_1 & E_1 \\ X_2 & Y_2 & Z_2 & E_2 \\ X_3 & Y_3 & Z_3 & E_3 \\ X_4 & Y_4 & Z_4 & E_4 \end{bmatrix}$$

$\det(A) = E_1 A_{14} - E_2 A_{24} + E_3 A_{34} - E_4 A_{44}$ (equal to zero if X,Y,Z,E are coplanar)

The point E resides on the plane defined by the points X,Y,Z iff det(A)=0

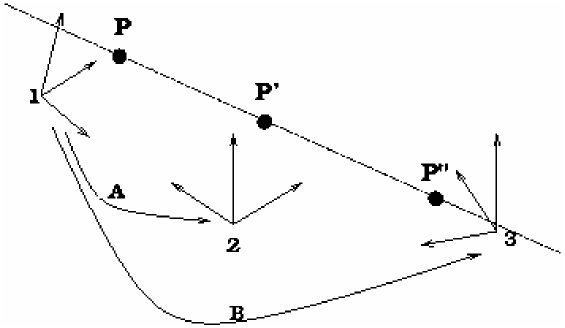
The plane (the dual coordinates) is represented by the vector $W = (w_1, w_2, w_3, w_4)$

$$w_1 = \det \begin{pmatrix} x_2 y_2 z_2 \\ x_3 y_3 z_3 \\ x_4 y_4 z_4 \end{pmatrix} \quad w_2 = -\det \begin{pmatrix} x_1 y_1 z_1 \\ x_3 y_3 z_3 \\ x_4 y_4 z_4 \end{pmatrix} \quad w_3 = \det \begin{pmatrix} x_1 y_1 z_1 \\ x_2 y_2 z_2 \\ x_4 y_4 z_4 \end{pmatrix} \quad w_4 = -\det \begin{pmatrix} x_1 y_1 z_1 \\ x_2 y_2 z_2 \\ x_3 y_3 z_3 \end{pmatrix}$$

$$w_i = \varepsilon_{ijkl} x^j y^k z^l$$

Where the entries of cross-product tensor ε consist of +1,-1,0 in the appropriate locations.

Homography Tensors of P^3



Basic constraint: $rank\begin{pmatrix} | & | & | \\ P & AP' & BP'' \\ | & | & | \end{pmatrix} = 2$

→ For every vector V: $\det\begin{pmatrix} | & | & | & | \\ P & AP' & BP'' & V \\ | & | & | & | \end{pmatrix} = 0$

→ $P^i(\varepsilon_{ilmu}(a^l_j P'^j)(b^m_k P''^k)v^u)$

$$= P^i P'^j P''^k(\varepsilon_{ilmu} a^l_j b^m_k v^u) = 0$$

Resulting in a 4x4x4 tensor $J_{ijk} = \varepsilon_{ilmu} a^l_j b^m_k v^u$ with the constraint $P^i P'^j P''^k J_{ijk} = 0$

Each matching triplet P,P',P'' arising from a dynamic point contributes one linear equation $P^i P'^j P''^k J_{ijk} = 0$ to a 4x4x4 tensor J. Any $N \geq 60$ matching triplets in general position provide an estimation matrix for J with a four-dimensional null space. The 60 points should be distributed along at least 10 lines, five of which can hold up to 8 dynamic points, and the remaining five up to 4 dynamic points.

Mixed Labeled and Unlabeled Static Points

Unlabeled Stationary Points:

The constraints $P^i P'^j P''^k J_{ijk} = 0$ made solely from unlabeled stationary points span at most a 20-dimensional space.

The 3-fold symmetric powers $Sym^3 V$ of a 4-dim vector space V is 20

$$rank(v_1 \otimes v_2 \otimes v_3 \mid rankspan(v_1, v_2, v_3) = 1) = \binom{n+3-1}{3} = \binom{4+3-1}{3} = 20$$

Mixed Labeled and Unlabeled Static Points

Labeled Stationary Points:

A labeled stationary point can provide at most 10 linearly independent constraints.

$$\det\begin{pmatrix} | & | & | & | \\ P & AP'B & \delta & V \\ | & | & | & | \end{pmatrix} = 0 \qquad \forall \delta, V$$

$$\begin{array}{lll} P^i P'^j e_1^k J_{ijk} = 0 & P^i e_1^j P''^k J_{ijk} = 0 & e_1^i P'^j P''^k J_{ijk} = 0 \\ P^i P'^j e_2^k J_{ijk} = 0 & P^i e_2^j P''^k J_{ijk} = 0 & e_2^i P'^j P''^k J_{ijk} = 0 \\ P^i P'^j e_3^k J_{ijk} = 0 & P^i e_3^j P''^k J_{ijk} = 0 & e_3^i P'^j P''^k J_{ijk} = 0 \\ P^i P'^j e_4^k J_{ijk} = 0 & P^i e_4^j P''^k J_{ijk} = 0 & e_4^i P'^j P''^k J_{ijk} = 0 \end{array}$$

Where e_1, e_2, e_3, e_4 are the standard basis $(1,0,0,0),...,(0,0,0,1)$

The constraint $P^i P'^j P''^k J_{ijk} = 0$ is accounted for 3 times (is spanned by each column).

Mixed Labeled and Unlabeled Static Points

Out of the ten linearly independent constraints arising from a labeled stationary point, 4 lie in the rank-20 subspace spanned by unlabeled stationary points and 6 lie in the subspace spanned only by dynamic points.

$P^i P'^j e_1^k J_{ijk} = 0$	$P^i e_1^j P''^k J_{ijk} = 0$	$e_1^i P'^j P''^k J_{ijk} = 0$
$P^i P'^j e_2^k J_{ijk} = 0$	$P^i e_2^j P''^k J_{ijk} = 0$	$e_2^i P'^j P''^k J_{ijk} = 0$
$P^i P'^j e_3^k J_{ijk} = 0$	$P^i e_3^j P''^k J_{ijk} = 0$	$e_3^i P'^j P''^k J_{ijk} = 0$
$P^i P'^j e_4^k J_{ijk} = 0$	$P^i e_4^j P''^k J_{ijk} = 0$	$e_4^i P'^j P''^k J_{ijk} = 0$

Add the three constraints of the first row: $G^{ijk} = P^i P^j e_1^k + P^i e_1^j P^k + e_1^k P^j P^k$

Then, G^{ijk} is a symmetric tensor and is therefore spanned by the 20-dim subspace of the unlabeled stationary points. Likewise for each of the remaining rows.

Mixed Labeled and Unlabeled Static Points

Corollary 1:

A minimum of 7 labeled stationary points are necessary for a unique (up to a 4-dimensional solution space) solution for J.

The first 5 labeled points will fill up the 20-dim subspace of unlabeled stationary points.
Each additional point can contribute at most 6 linearly independent constraints.
$5 \times 10 + 2 \times 6 >= 60$.

Note: A stationary 3D-3D alignment requires only 5 matching points !

Mixed Labeled and Unlabeled Static Points

Corollary 2:

In a situation of matching triplets arising from a mixture of stationary and moving points, let x<= 7 be the number of matching triplets that are known a priori to arise from stationary points. To obtain a unique linear solution for J (up to a 4-dimensional solution space), the minimal number of unlabeled matching triplets required is:

$$\left\{ \begin{array}{ll} 60 - 10x & x \leq 5 \\ 4 & x = 6 \\ 0 & x = 7 \end{array} \right\},$$

out of which 40-6x, x<7, should be dynamic and at most 20-4x, x<= 5, could be unlabeled stationary points.

Tensor Slices and the Extraction of A,B

Note that the tensor J is recovered up to a 4-dim subspace of solutions, i.e., from the measurements we have 4 tensors J^1, J^2, J^3, J^4 each associated with its own different V.

Recovering the "principal point" V:

Consider the plane π defined by the points $V,\ AP',\ BP''$

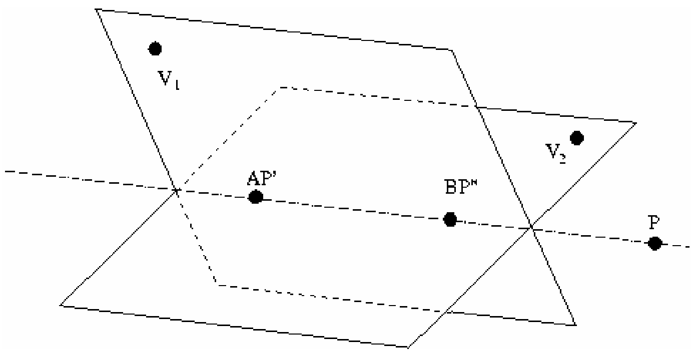
$$\pi_i = P'^j P''^k J_{ijk} = \varepsilon_{ilmu}(a^l_j P'^j)(b^m_k P''^k)v^u$$

By varying P',P'' we obtain a star of planes all coincident with V. Thus, V can be recovered By taking 3 double slices of the tensor and finding their intersection point.

Recovering the trajectory line between AP' and BP''

Take any two tensors (out of J^1, J^2, J^3, J^4) and find the intersection:

$$P'^j P''^k J^1_{ijk} \cap P'^j P''^k J^2_{ijk}$$



Tensor Slices and the Extraction of A,B

Use a single contraction to recover A:

A single contraction $H_{ij} = P''^{k} J_{ijk}$ is a 4x4 matrix H that maps points to planes.
The plane HP', i.e., $P'^{j} H_{ij} = P'^{j} P''^{k} J_{ijk}$ contains the points V,AP',BP''

→ The range of the mapping induced by H contains the line passing through V,BP'' thus rank(H)=2.

Because HP' is the plane through V,AP',BP'', we must have $P'^{\top} A^{\top} H P' = 0$ for all P'

→ $A^{\top} H$ is skew-symmetric and thus provides 10 linear constraints on A.

By varying P'' and using one of the additional tensors we can obtain sufficient constraints
To solve for A.

- 4x4x4
- 4 Solutions (60 matching points)
- Unlabeled Static points live in a 20-dim subspace.
- Labeled static point contributes 10 constraints
- 4 of the 10 constraints provided by a labeled static live in the 20'th dimensional subspace of unlabeled static points.

Segmentation Results

The General Case

- N: dim of vector space (3 for Htensor, 4 for Jtensor)
- K: No. of observations (3 for Htensor, 3 for Jtensor)
- M: dim of subspace (2 for Htensor, 2 for Jtensor)

<332> Htensor moving points, 26 dim-space
<331> Htensor static points, 10 dim-space
<432> Jtensor moving points, 60 dim-space
<431> Jtensor static points, 20 dim-space

<nkm> ?

Let V be a n-dim vector space. For $n \geq k \geq m$

Consider the $GL(V)$ module $V(n,m,k)$

$$V(n,m,k) = \{v_1 \otimes v_2 \cdots \otimes v_m \mid \dim span\{v_1,\ldots,v_m\} \leq k\}$$

We want to determine the module structure and $\dim V(n,m,k)$

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition of m

$$\dim V(n,m,k) = n^m - \sum_{\lambda_{k+1}>0} \frac{m!}{\prod (hook-lengths)} \frac{\prod_{(i,j)\in\lambda} (n+j-i)}{\prod (hook-lengths)}$$

$$V \otimes V = Sym^2 V \oplus \wedge^2 V$$

This is not true for $V^{\otimes m},\ m > 2$

$f_\lambda = \frac{m!}{\prod_{(i,j)} h_{ij}}$ the number of standard tableaux on λ

The hook-length $h_{ij} = \lambda_i + \mu_j - i - j + 1$

$d_\lambda(n) = \prod_{(i,j)} \frac{n - i + j}{h_{ij}}$ The number of semi-standard tableaux (number of ways to fill The diagram with numbers 1,…,n such that all rows are Non-decreasing and all columns are increasing

$$\dim V^{\otimes m} = n^m = \sum_{\lambda \vdash m} d_\lambda(n) f_\lambda.$$

$$\dim V(n,m,k) = n^m - \sum_{\lambda_{k+1} \neq 0} f_\lambda d_\lambda(n)$$

$$V(n,k,k-1) = n^k - \binom{n}{k}$$

$$V(3,3,2) = 3^3 - \binom{3}{3} = 26$$ Htensor

$$V(4,3,2) = 4^3 - \binom{4}{3} = 60$$ Jtensor

$$V(n,k,k-2) = n^k - \left[\binom{n}{k} + (k-1)^2 \binom{n+1}{k}\right]$$

$$V(3,3,1) = 3^3 - \left[\binom{3}{3} + (3-1)^2 \binom{3+1}{3}\right] = 27 - 17 = 10$$ Htensor, static

$$V(4,3,1) = 4^3 - \left[\binom{4}{3} + (3-1)^2 \binom{4+1}{3}\right] = 64 - 44 = 20$$ Jtensor, static

Open Problems

1. Is the dimension of V(n,m,k) sufficient for uniquely recovering the m-1 individual colineations?
   for example, we saw that A,B can uniquely be recovered from 4x4x4 J even though J cannot be uniquely recovered from measurements (rank of measurements at most 60).

2. What are the constraints contributed from a labeled k’ < k-dimensional moving point?
   for example, we saw that a stationary (k’=1) point for V(3,3,2) contribute 7 constraints and 10 for V(4,3,2).

3. What would be the dimension of the space covered by mixed observations? I.e. from labeled k’ < k-dim labeled, and unlabeled k’’ < k.
   for example, we saw that labeled stationary (k’=1) for V(3,3,2) provide only 4 new constraints as 3 out of the 7 constraints are contained in unlabeled stationary points.