Multi-linear Systems for 3D-from-2D Interpretation

Lecture 2

Duality and Shape Tensors

Amnon Shashua

Hebrew University of Jerusalem
Israel

Material We Will Cover Today

- The Point-View Duality Principle
- Multi-point Constraints of a single view
- Single-View Tensors
- Properties of Single View Tensors
- Single View Tensors under a stabilized reference plane

Weinshall,Werman,Shashua ECCV 1996
Carlsson, ECCV 1996
Carlsson,Weinshall IJCV 1998

Principle of duality

Irani,Anandan,Weinshall ECCV 98

Duality using a reference plane

Levin, Shashua ECCV 2002

Underlying tensors, completing the story

Point-View Duality Principle

$$P_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \\ W_i \end{pmatrix}$$ points in 3D projective space

$$p_i \cong MP_i$$ points in 2D projective space

Approach: with enough points, we can "eliminate" M

Point-View Duality Principle

We are free to choose a projective basis for the 3D space, and we fix 12 degrees of freedom (out of 16):

$$p_i \cong MP_i = MT^{-1}TP_i$$

Let us choose a representation of 3D space such that the first 4 points have the "standard" coordinates:

$$P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad P_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

each point fixes 3 degrees of freedom

Point-View Duality Principle

We can also choose a basis for the 2D (image) projective space:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cong M \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cong M \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cong M \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cong M \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Point-View Duality Principle

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \cong M \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \Rightarrow M_1 = \begin{pmatrix}\alpha\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \cong M \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \Rightarrow M_2 = \begin{pmatrix}0\\\beta\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \cong M \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \Rightarrow M_3 = \begin{pmatrix}0\\0\\\gamma\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \cong M \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \Rightarrow M_4 = \begin{pmatrix}\delta\\\delta\\\delta\end{pmatrix}$$

$$M = \begin{bmatrix}\alpha & 0 & 0 & \delta\\ 0 & \beta & 0 & \delta\\ 0 & 0 & \gamma & \delta\end{bmatrix}$$

Point-View Duality Principle

$$M = \begin{bmatrix} \alpha & 0 & 0 & \delta \\ 0 & \beta & 0 & \delta \\ 0 & 0 & \gamma & \delta \end{bmatrix}$$

Note: in the new basis, M is represented by 4 numbers (!)

Note: in the new basis we cannot assume that

$$p \cong \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

(third coordinate may vanish), thus

$$p \cong \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Point-View Duality Principle

for $\quad i = 5,6,7....$

$$p_i \cong \begin{bmatrix} \alpha & 0 & 0 & \delta \\ 0 & \beta & 0 & \delta \\ 0 & 0 & \gamma & \delta \end{bmatrix} \begin{pmatrix} X_i \\ Y_i \\ Z_i \\ W_i \end{pmatrix} = \begin{bmatrix} X_i & 0 & 0 & W_i \\ 0 & Y_i & 0 & W_i \\ 0 & 0 & Z_i & W_i \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

$$\boxed{p_i \cong MP_i = \hat{P}_i \hat{M}}$$

$$\hat{M} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \qquad null(M) = \begin{pmatrix} 1/\alpha \\ 1/\beta \\ 1/\gamma \\ -1/\delta \end{pmatrix}$$

Single-view Multilinear Constraints

Let l_i, l_i' be two lines coincident with p_i

$$l_i^T p_i = 0 \qquad l_i'^T p_i = 0$$

$$p_i \cong \hat{P}_i \hat{M} \quad \Longrightarrow \quad \begin{aligned} l_i^T \hat{P}_i \hat{M} &= 0 \\ l_i'^T \hat{P}_i \hat{M} &= 0 \end{aligned}$$

Single-view Multilinear Constraints

for $\quad i = 5,6,7,8$

$$\begin{bmatrix} l_5^T \hat{P}_5 \\ \cdot \\ l_8^T \hat{P}_8 \\ l_5^{'T} \hat{P}_5 \\ \cdot \\ l_8^{'T} \hat{P}_8 \end{bmatrix}_{8\times 4} \hat{M} = 0$$

Every 4x4 minor vanishes !

The choice of 4 rows can include 2,3,4 points
(in addition to the 4 basis points)

Single-view Bilinear Constraints

Choose the 4 rows such that they include only 2 points:

$$\det \begin{bmatrix} l_5^{T} \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^{T} \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \end{bmatrix} = 0$$

Single-view Bilinear Constraints

$$\det \begin{bmatrix} l_5^T \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^T \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \end{bmatrix} = 0$$

We still have 4 d.o.f left, so we can set the coordinates of

$$P_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Let $P_6 = \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}$

$$p \cong \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longrightarrow l \cong \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \qquad l' \cong \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}$$

Single-view Bilinear Constraints

$$\det \begin{bmatrix} l_5^T \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^T \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \end{bmatrix} = 0$$

$$\hat{P}_5 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \hat{P}_6 = \begin{bmatrix} X & 0 & 0 & W \\ 0 & Y & 0 & W \\ 0 & 0 & Z & W \end{bmatrix}$$

$$l \cong \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \qquad l' \cong \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\det \begin{bmatrix} -z_5 & 0 & x_5 & x_5 - z_5 \\ 0 & -z_5 & y_5 & y_5 - z_5 \\ -z_6 X & 0 & x_6 Z & x_6 W - z_6 W \\ 0 & -z_6 Y & y_6 Z & y_6 W - z_6 W \end{bmatrix} = 0}$$

Single-view Bilinear Constraints

$$\det\begin{bmatrix} -z_5 & 0 & x_5 & x_5 - z_5 \\ 0 & -z_5 & y_5 & y_5 - z_5 \\ -z_6 X & 0 & x_6 Z & x_6 W - z_6 W \\ 0 & -z_6 Y & y_6 Z & y_6 W - z_6 W \end{bmatrix} = 0$$

$$p_6^T G p_5 = 0$$

for all viewing positions !

p_5, p_6 change with viewing position

$$G = \begin{bmatrix} 0 & WY - YW & YZ - WZ \\ XZ - XW & 0 & -XZ + WZ \\ -XY + WX & XY - WY & 0 \end{bmatrix}$$

Single-view Bilinear Constraints

$$p_6^T G p_5 = 0 \qquad G = \begin{bmatrix} 0 & WY - YW & YZ - WZ \\ XZ - XW & 0 & -XZ + WZ \\ -XY + WX & XY - WY & 0 \end{bmatrix}$$

$$G_{11} = G_{22} = G_{33} = 0 \qquad \sum_{ij} G_{ij} = 0$$

$$\det(G) = 0$$

The dual fundamental matrix has additional 4 linear constraints!
(compared to the fundamental matrix). See next slides for a proof.

The Dual Fundamental Matrix

Why $G_{11} = G_{22} = G_{33} = 0$ $\sum_{ij} G_{ij} = 0$?

Unlike the multiview tensors, the single-view point tensors have internal linear constraints ("synthetic" constraints) which are related to the fact the dual projection matrices are sparse.

Since $p_6^T G p_5 = 0$ holds for all camera matrices,

consider the camera matrices M_j

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e_1 e_2 e_3 e_4

The Dual Fundamental Matrix

Why $G_{11} = G_{22} = G_{33} = 0$ $\sum_{ij} G_{ij} = 0$?

$M_j P \cong \{e_j \quad 0\} \quad \forall P$ $\Rightarrow$ $l^T \hat{P} \hat{M}_j = 0 \quad \forall \hat{P}$

for all $l^T e_j = 0$

$\Rightarrow$ $\det \begin{bmatrix} l_5^T \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^T \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \end{bmatrix} = 0$

$l_5, l_5^{'}, l_6, l_6^{'}$ coincident with e_j

holds for all choices of $\hat{P}_5, \hat{P}_6$!!

$\Rightarrow$ $e_j^T G e_j = 0 \qquad j = 1, \ldots, 4$

The Dual Fundamental Matrix

Why $\det(G) = 0$?

$p_6^T G p_5 = 0$ $\Longrightarrow$ $G p_5$ is a line coincident with p_6

Since $p_6^T G p_5 = 0$ holds for all camera matrices,
choose the camera projection matrix such that $MP_6 = 0$
that is, the center of projection is at P_6

$\Longrightarrow$ $l^T M P_6 = 0 \quad \forall l$ $\Longrightarrow$ $l^T \hat{P}_6 \hat{M} = 0$

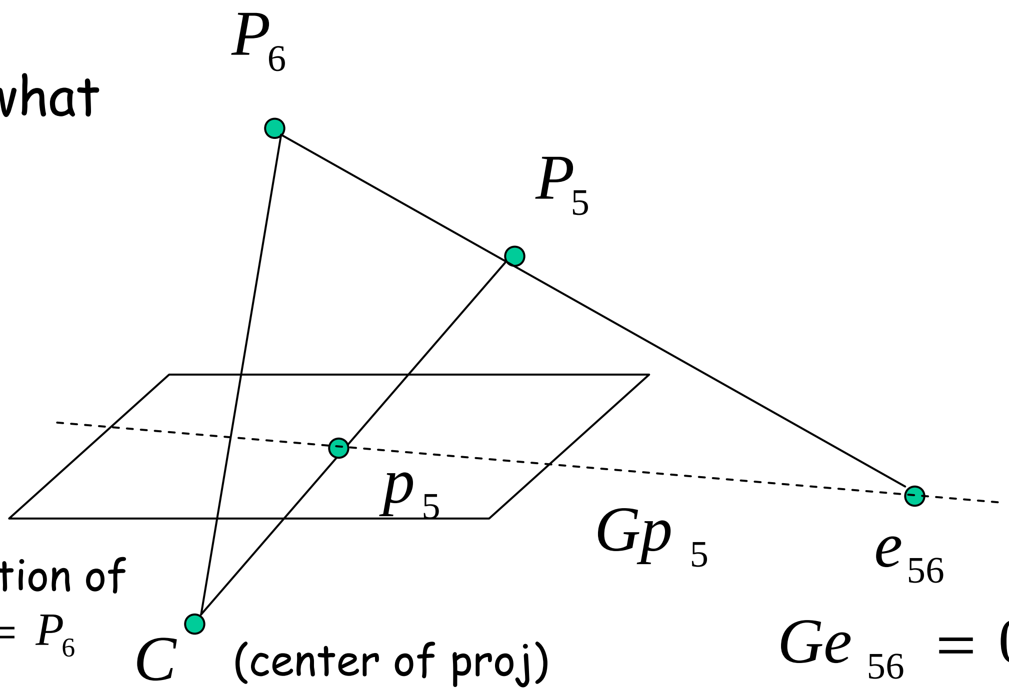
$\Longrightarrow$ $\begin{bmatrix} l_5^T \hat{P}_5 \\ l_5'^T \hat{P}_5 \\ l^T \hat{P}_6 \\ l'^T \hat{P}_6 \end{bmatrix} \hat{M} = 0 \quad \forall l, l'$ $\Longrightarrow$ $p^T G p_5 = 0 \quad \forall p$

$\Longrightarrow$ $G p_5 = 0$ $\Longrightarrow$ $\det(G) = 0$

Note: p_5 in this situation is the projection of P_5 when $null(M) = P_6$
we will denote this point by $p_5 = e_{56}$

A "sketch" – the sketch is somewhat misleading since there are two (dual) epipoles.

Note: while C changes so does the image plane!



$Ge_{56} = 0$

Claim:

$e_{56} = \hat{P}_5 null(\hat{P}_6)$ is the projection of P_5 when $C = P_6$

Proof:

$$\underbrace{\begin{bmatrix} X_5 & 0 & 0 & W_5 \\ 0 & Y_5 & 0 & W_5 \\ 0 & 0 & Z_5 & W_5 \end{bmatrix} \begin{pmatrix} 1/X_6 \\ 1/Y_6 \\ 1/Z_6 \\ -1/W_6 \end{pmatrix}}_{\hat{P}_5 null(\hat{P}_6)} = \underbrace{\begin{bmatrix} 1/X_6 & 0 & 0 & -1/W_6 \\ 0 & 1/Y_6 & 0 & -1/W_6 \\ 0 & 0 & 1/Z_6 & -1/W_6 \end{bmatrix} \begin{pmatrix} X_5 \\ Y_5 \\ Z_5 \\ W_5 \end{pmatrix}}_{\text{projection of } P_5 \text{ when } null(M) = P_6}$$

$G^T e_{65} = 0$

$e_{65} = \hat{P}_6 null(\hat{P}_5)$ is the projection of P_6 when $C = P_5$

6-point Single-view Indexing

$p_6^T G p_5 = 0$ is a function of 6 points (4 basis, 2 additional)

Each view provides one constraint for G. How many views are necessary?

G is defined up to scale and has 4 linear constraints – thus there are 4 parameters to solve for.

For a linear solution: 4 views are necessary.
For non-linear: 3 views are necessary (3-fold ambiguity).

Once G is recovered, the function $p_6^T G p_5 = 0$ is view-independent

thus provides a connection between 6 image points and their corresponding 3D points which does not depend on the point of view (indexing function).

7-point Single-view Tensor

12 choices to choose 4 rows which include
7 points (4 basis +3).
there are 3 groups: choose 2 rows from a single point and the remaining two rows one from each remaining point.

$$0 = \det \begin{bmatrix} l_5^T \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^T \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \\ l_7^T \hat{P}_7 \\ l_7^{'T} \hat{P}_7 \end{bmatrix}$$

choose these two rows

choose 1 rows from here

choose 1 rows from here

→ 4 trilinear constraints (view independent) (per group)

7-point Single-view Tensor

$$0 = \det \begin{bmatrix} l_5^T \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^T \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \\ l_7^T \hat{P}_7 \\ l_7^{'T} \hat{P}_7 \end{bmatrix} \Rightarrow \quad p_5^i l_j^6 l_k^7 T_i^{jk} = 0$$

l^6 is some line incident with p_6

l^7 is some line incident with p_7

T_i^{jk} is a trilinear function of P_5, P_6, P_7 alone.

7-point Single-view Tensor

The "synthetic" constraints for the dual trilinear tensor:

Since $p_5^i l_j^6 l_k^7 T_i^{jk} = 0$ holds for all camera matrices,

consider the camera matrices M_j

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e_1 e_2 e_3 e_4

$M_j P \cong \{e_j \quad 0\} \quad \forall P$ ⟹ $l^T \hat{P} \hat{M}_j = 0 \quad \forall \hat{P}$ for all $l^T e_j = 0$

$p_5^i l_j l'_k T_i^{jk} = 0$ where l, l' incident with $e = e_j$

⟹ $e^i l_j l'_k T_i^{jk} = 0$

4 constraints (2 choices for each line l, l')

⟹ 16 synthetic constraints (4 for each $e = e_j$)

7-point Single-view Tensor

The 7-point tensor is defined by 11 parameters (up to scale).

→ A linear solution will requires 3 views (each view 4 constraints)

How many Non-linear constraints?

The number of parameters needed to describe a 7-point configuration Is 3+3=6 (because the first 5 points can be assigned the standard coordinates and the remaining two points contribute 3 parameters each).

→ There should be 5 non-linear constraints (1 for the overall scale, and 4 additional).

7-point Single-view Tensor

The dual reprojection equation:

$$\boxed{p_5^i l_j^6 T_i^{jk} \cong p_7^k}$$

Given point p_5 and a line l^6 coincident with p_6

then the contraction $p_5^i l_j^6 T_i^{jk}$ produces the point p_7

6 image points and the shape tensor determine the 7th point

8-point Single-view Tensor

$$0 = \det \begin{bmatrix} l_5^T \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^T \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \\ l_7^T \hat{P}_7 \\ l_7^{'T} \hat{P} \\ l_8^T \hat{P}_8 \\ l_8^{'T} \hat{P}_{8\,7} \end{bmatrix} \Longrightarrow l_i^5 l_j^6 l_k^7 l_t^8 Q^{ijkt} = 0$$

l^6 is some line incident with p_6

l^7 is some line incident with p_7

l^8 is some line incident with p_8

Q^{ijkt} is a quadlinear function of P_5, P_6, P_7, P_8 alone.

8-point Single-view Tensor: Synthetic Constraints

Since $l_i^5 l_j^6 l_k^7 l_t^8 Q^{ijkt} = 0$ holds for all camera matrices,

consider the camera matrices M_j

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e_1 \qquad e_2 \qquad e_3 \qquad e_4$$

$M_j P \cong \{e_j \quad 0\} \quad \forall P$ ⟹ $l^T \hat{P} \hat{M}_j = 0 \quad \forall \hat{P}$ for all $l^T e_j = 0$

$l_i^5 l_j^6 l_k^7 l_t^8 Q^{ijkt} = 0$ where all 4 lines are incident with $e = e_j$

⟹ the constraint holds for all quadlinear tensors, 16 constraints

16 constraints from e_1, 15 from e_2, 14 from e_3, and 13 from e_4

⟹ 58 synthetic constraints.

8-point Single-view Tensor: How Many Views?

The 8-point tensor is defined by 81-58=23 parameters (up to scale).

View 1: 12 constraints since the lines through p_5, p_6, p_7, p_8 passing through e_1, e_2, e_3, e_4 are already accounted for by the synthetic constraints.

View 2: 11 independent constraints.

→ Two views are sufficient to recover the dual quadtensor

Duality Under a Fixed Reference Plane

Assume some plane has been identified and “stabilized” throughout the sequence of images (i.e., points on the reference plane project to fixed points throughout the sequence).

The house façade is stabilized

The road sign is stabilized

(movie clips due to Michal Irani)

Duality Under a Fixed Reference Plane

In this case, the first 4 basis points are coplanar:

$$P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad P_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad P_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Duality Under a Fixed Reference Plane

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \cong M\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \Rightarrow M_1 \cong \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \cong M\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \Rightarrow M_2 \cong \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \cong M\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \Rightarrow M_3 \cong \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \cong M\begin{pmatrix}1\\1\\1\\0\end{pmatrix} \Rightarrow M_1 + M_2 + M_3 \cong \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$M = \begin{bmatrix}\delta & 0 & 0 & \alpha\\ 0 & \delta & 0 & \beta\\ 0 & 0 & \delta & \gamma\end{bmatrix}$$

$$M_4 = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

(unconstrained)

Duality Under a Fixed Reference Plane

for $i = 5,6,7....$

$$p_i \cong \begin{bmatrix} \delta & 0 & 0 & \alpha \\ 0 & \delta & 0 & \beta \\ 0 & 0 & \delta & \gamma \end{bmatrix} \begin{pmatrix} X_i \\ Y_i \\ Z_i \\ W_i \end{pmatrix} = \begin{bmatrix} W_i & 0 & 0 & X_i \\ 0 & W_i & 0 & Y_i \\ 0 & 0 & W_i & Z_i \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

$$\boxed{p_i \cong MP_i = \hat{P}_i \hat{M}}$$

$$\hat{M} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \qquad null(M) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ -\delta \end{pmatrix}$$

Duality Under a Fixed Reference Plane

$$\hat{M} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \qquad null(M) = \begin{pmatrix} 1/\alpha \\ 1/\beta \\ 1/\gamma \\ -1/\delta \end{pmatrix} \qquad null(M) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ -\delta \end{pmatrix}$$

(general)

(ref plane stabilized)

When $\hat{M}$ varies along a linear subspace (say a line)	$null(M)$ varies along an algebraic surface	$null(M)$ varies along a linear subspace of the same dimension (easy geometric interp)

6-point Tensor Under a Fixed Reference Plane

$$\det \begin{bmatrix} l_5^T \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^T \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \end{bmatrix} = 0$$

$$\hat{P}_5 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \hat{P}_6 = \begin{bmatrix} W & 0 & 0 & X \\ 0 & W & 0 & Y \\ 0 & 0 & W & Z \end{bmatrix}$$

$$p_6^T G p_5 = 0$$

What is different?

6-point Tensor Under a Fixed Reference Plane

$$p_6^T G p_5 = 0$$

Since P_1, P_2, P_3, P_4 are coplanar

$P_i^T n = 0$ where $n = (0,0,0,1)^T$

Since the bilinear constraint holds for all choices of M

consider: $M = un^T = \begin{bmatrix} 0 & 0 & 0 & u_1 \\ 0 & 0 & 0 & u_2 \\ 0 & 0 & 0 & u_3 \end{bmatrix} \Rightarrow MP \cong \{u \quad 0\} \quad \forall P$

$$\det \begin{bmatrix} l_5^T \hat{P}_5 \\ l_5^{'T} \hat{P}_5 \\ l_6^T \hat{P}_6 \\ l_6^{'T} \hat{P}_6 \end{bmatrix} = 0$$

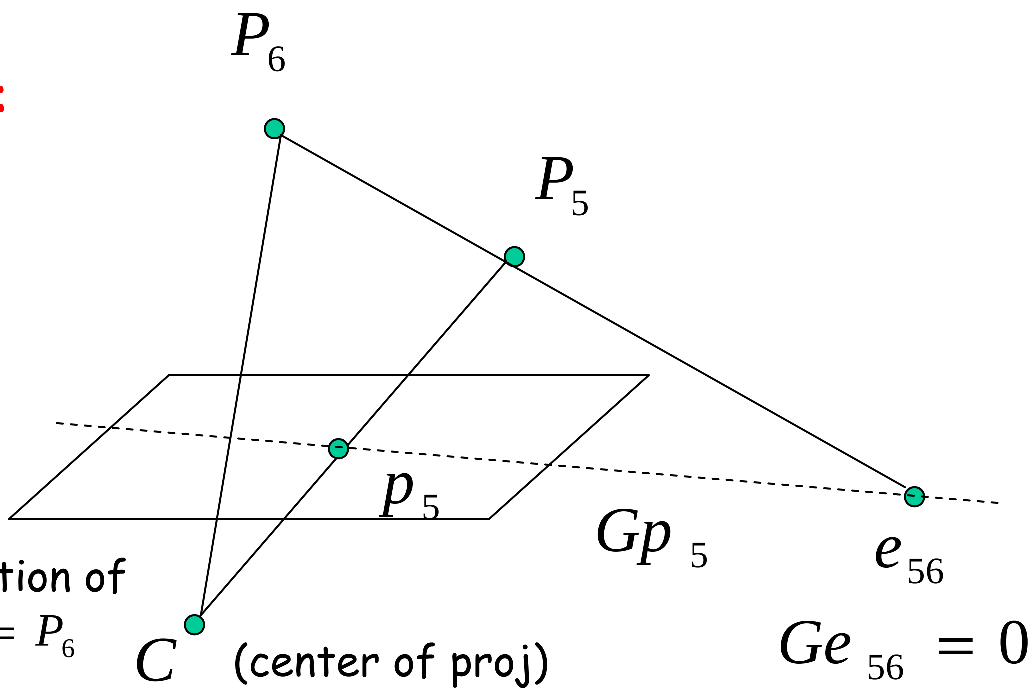
l_5, l_5', l_6, l_6' coincident with u

holds for all choices of $\hat{P}_5, \hat{P}_6$!!

$\Rightarrow$ $u^T G u = 0$

$\Rightarrow$ G is skew-symmetric (only 3 parameters)

Reminder from the general case:



Claim:

$e_{56} = \hat{P}_5 \, null(\hat{P}_6)$ is the projection of P_5 when $C = P_6$

Proof:

$$\underbrace{\begin{bmatrix} X_5 & 0 & 0 & W_5 \\ 0 & Y_5 & 0 & W_5 \\ 0 & 0 & Z_5 & W_5 \end{bmatrix} \begin{pmatrix} 1/X_6 \\ 1/Y_6 \\ 1/Z_6 \\ -1/W_6 \end{pmatrix}}_{\hat{P}_5 \, null(\hat{P}_6)} = \underbrace{\begin{bmatrix} 1/X_6 & 0 & 0 & -1/W_6 \\ 0 & 1/Y_6 & 0 & -1/W_6 \\ 0 & 0 & 1/Z_6 & -1/W_6 \end{bmatrix} \begin{pmatrix} X_5 \\ Y_5 \\ Z_5 \\ W_5 \end{pmatrix}}_{\text{projection of } P_5 \text{ when } null(M) = P_6}$$

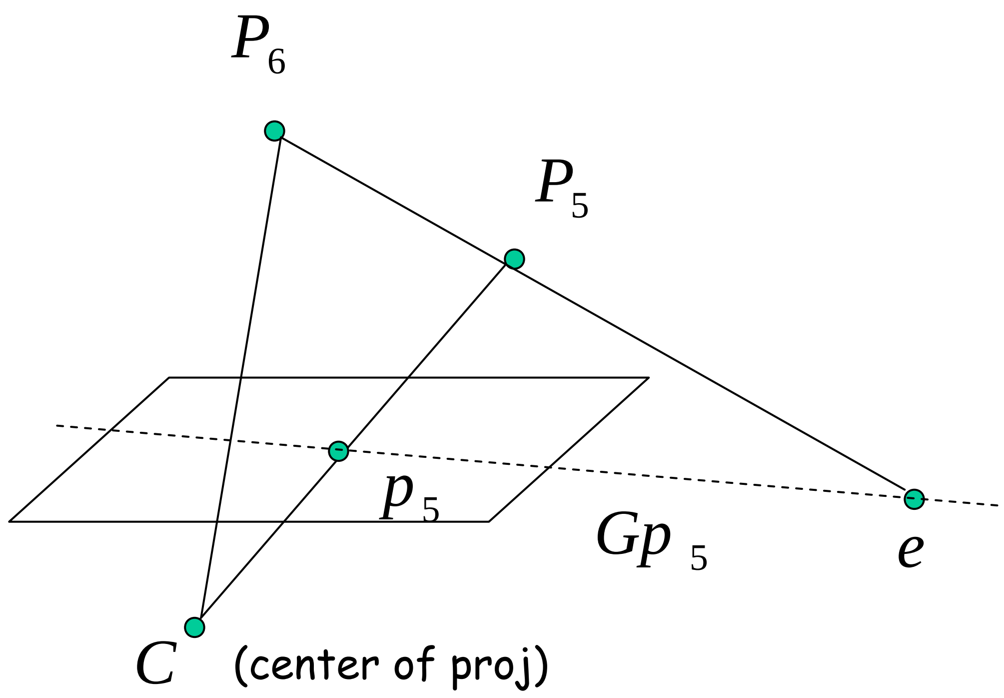
$$G^T e_{65} = 0$$

$e_{65} = \hat{P}_6 \, null(\hat{P}_5)$ is the projection of P_6 when $C = P_5$

Simple Geometric Interpretation

$$G = -G^T$$

$$\left.\begin{array}{l} Ge_{56} = 0 \\ G^T e_{65} = 0 \end{array}\right\} \quad e_{56} = e_{65}$$



There is only one (dual) epipole!

The Dual Image Ray

Recall: image ray in multi-view geometry is defined as follows:

Let l, l' be 2 lines coincident with $p \cong MP$

$$\left.\begin{aligned} l^T MP &= 0 \\ l'^T MP &= 0 \end{aligned}\right\}$$

P has a 1-parameter degree of freedom defined by the intersection of the two planes

$l^T M, l'^T M$ this is the image ray

(the lines passing through p and $null(M)$)

The Dual Image Ray

In Dual:

$$\left.\begin{aligned} l^T \hat{P} \hat{M} &= 0 \\ l'^T \hat{P} \hat{M} &= 0 \end{aligned}\right\}$$

$\hat{M}$ has a 1-parameter degree of freedom

The line passing through p and $null(\hat{P})$

In general case: the camera center varies along a non-linear curve!

i.e., when $\hat{M} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$ varies along a line, the camera center $null(M) = \begin{pmatrix} 1/\alpha \\ 1/\beta \\ 1/\gamma \\ -1/\delta \end{pmatrix}$

varies along a curve.

The Dual Image Ray

In ref-plane stabilized: the camera center varies along a line,

and the line is the one joining p and P

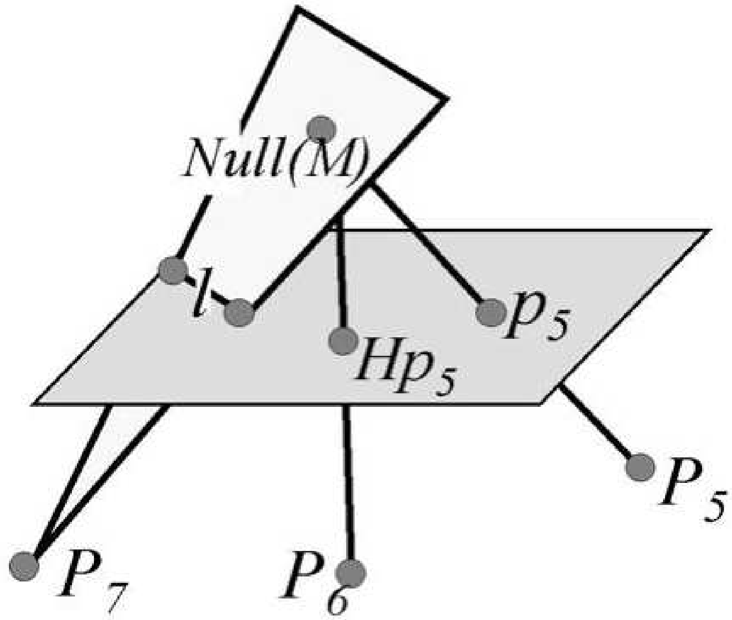
$$0 = l^T \hat{P} \hat{M} = l^T \begin{bmatrix} W & 0 & 0 & X \\ 0 & W & 0 & Y \\ 0 & 0 & W & Z \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = l^T \underbrace{\begin{bmatrix} -W & 0 & 0 & X \\ 0 & -W & 0 & Y \\ 0 & 0 & -W & Z \end{bmatrix}}_{null = P} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ -\delta \end{pmatrix}}_{null(M)}$$

$\Rightarrow$ $null(M)$ varies along the line $\overline{pP}$

Items we have Skipped

- Number of Synthetic Constraints for 7-point tensor is 21. There no non-linear constraints. Only 2 views are required to solve.

- 8-point dual tensor: 3x3x3x3 tensor, 72 synthetic constraints, no non-linear constraints. 2 views are required (first contributes 5 constraints, the second 4 constraints).


Null(M)
l
p5
Hp5
P5
P7
P6