Multi-linear Systems for 3D-from-2D Interpretation

Lecture 1

Multi-view Geometry from a Stationary Scene

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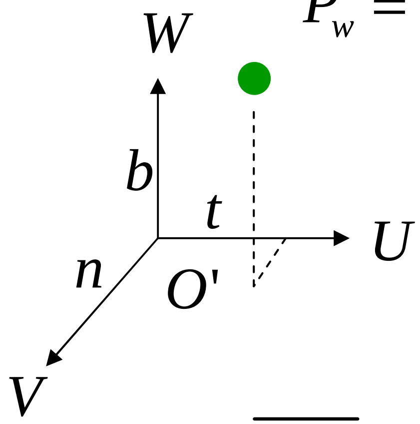
Material We Will Cover Today

- The structure of 3D->2D projection matrix
- A primer on projective geometry of the plane
- Epipolar Geometry and Fundamental Matrix
- Why 3 views?
- Primer on Covariant-Contravariant Index conventions
- Trifocal Tensor
- Quadrifocal Tensor

The structure of 3D->2D projection matrix

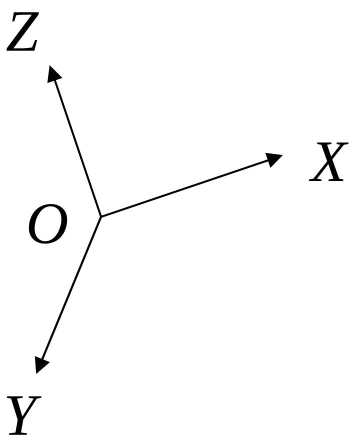
The Structure of a Projection Matrix

$$P_w = \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$



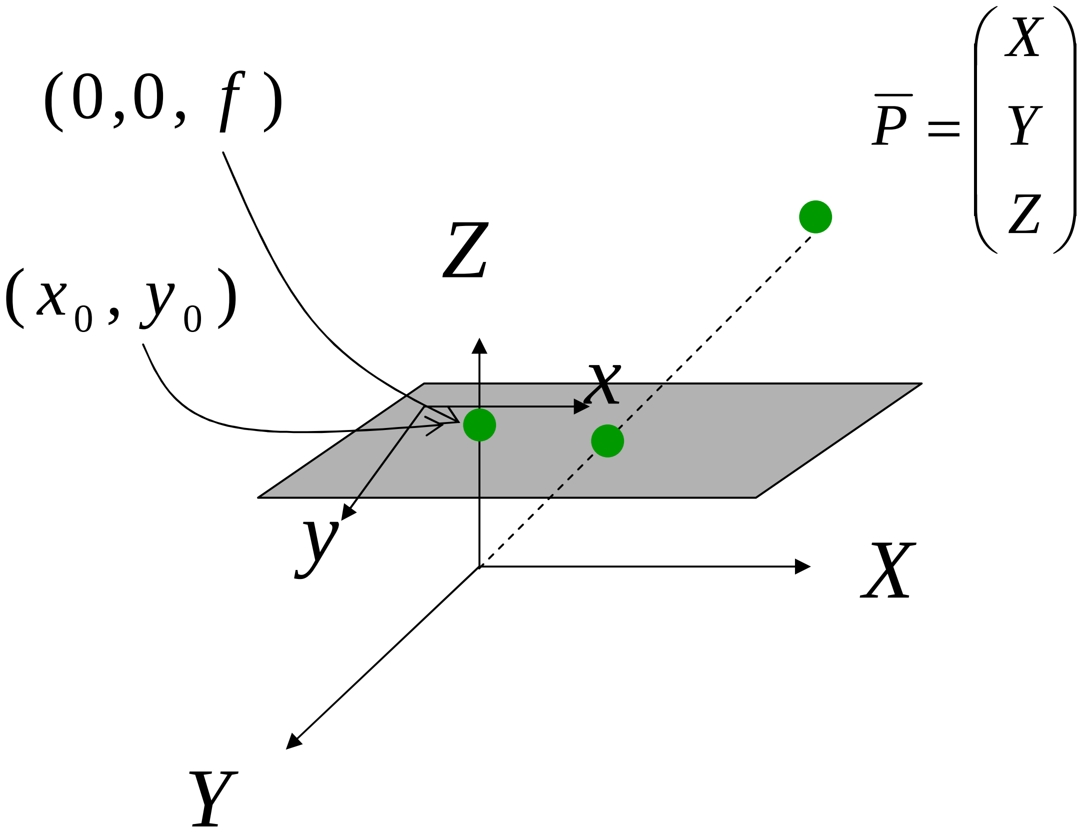
$$\overline{OP} = \overline{OO'} + Ut + Vn + Wb$$

$$R = \begin{bmatrix} t & n & b \end{bmatrix}$$



$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} U \\ V \\ W \end{pmatrix} + T$$

The Structure of a Projection Matrix



$$x = f\frac{X}{Z} + x_0$$

$$y = f\frac{Y}{Z} + y_0$$

The Structure of a Projection Matrix

$$x = f\frac{X}{Z} + x_0$$

$$y = f\frac{Y}{Z} + y_0$$

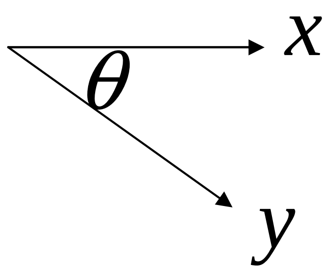
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{bmatrix} f & 0 & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = K(RP_w + T) = K[R;T] \begin{pmatrix} U \\ V \\ W \\ 1 \end{pmatrix}$$

$$p \cong M_{3\times 4} P$$

The Structure of a Projection Matrix

Generally,

$$K = \begin{bmatrix} f_x & f_x \dfrac{\cos\theta}{\sin\theta} & x_0 \\ 0 & \dfrac{f_y}{\sin\theta} & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$



(x_0, y_0) is called the "principle point"

$s = f_x \dfrac{\cos\theta}{\sin\theta}$ is called the "skew"

$\dfrac{f_y}{f_x}$ is "aspect ratio"

The Camera Center

$$p \cong M_{3\times 4} P$$

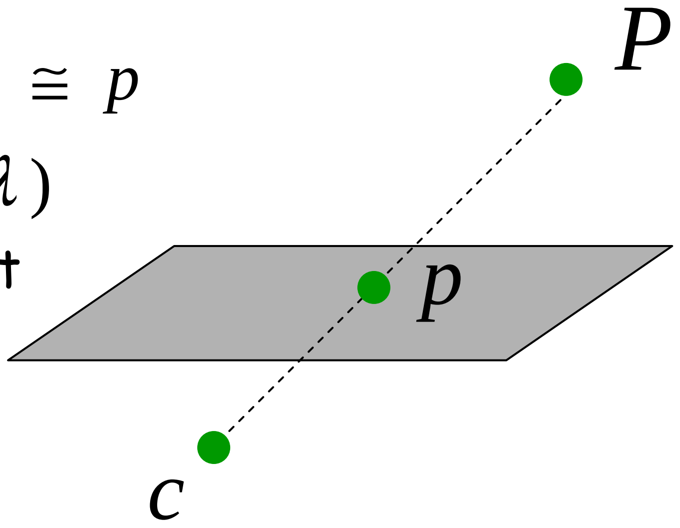
M has rank=3, thus $\exists c$ such that $Mc = 0$

Why is c the camera center?

Consider the “optical ray” $Q(\lambda) = \lambda P + (1 - \lambda)c$

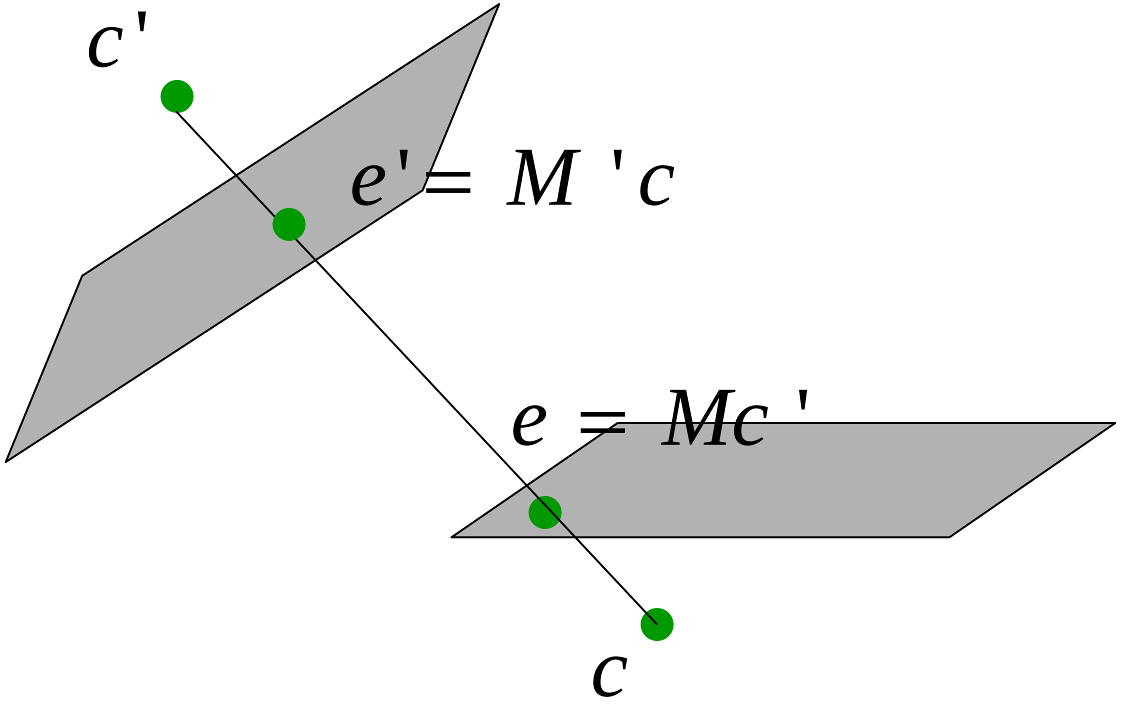
$$MQ(\lambda) = \lambda MP \cong MP \cong p$$

All points along the line $Q(\lambda)$ are mapped to the same point



➡ $Q(\lambda)$ is a ray through the camera center

The Epipolar Points



Choice of Canonical Frame

$$p \cong MP = MWW^{-1}P$$

$$p' \cong M'P = M'WW^{-1}P$$

$W^{-1}P$ is the new world coordinate frame

We have 15 degrees of freedom (16 upto scale)

Choose W such that $MW = [I;0]$

Choice of Canonical Frame

Let $$M = [\overline{M}; m]$$

$$MW = [\overline{M}; m] \begin{bmatrix} \overline{M}^{-1} - (\overline{M} m) n^T & -(1/\lambda)\overline{M}^{-1} m \\ n^T & 1/\lambda \end{bmatrix}$$

$$= \left[I - mn^T + mn^T; \quad -(1/\lambda) m + (1/\lambda) m \right]$$

$$= [I; \quad 0]$$

We are left with 4 degrees of freedom (upto scale): (n^T, λ)

Choice of Canonical Frame

$$p \cong [I;0]\overline{P} \qquad p = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \overline{P} = \begin{pmatrix} p \\ \mu' \end{pmatrix}$$

$$p' \cong [H;e']\overline{P}$$

$$p \cong [I;0]\begin{bmatrix} I & 0 \\ n^T & 1/\lambda \end{bmatrix}\begin{bmatrix} I & 0 \\ -\lambda n^T & \lambda \end{bmatrix}\overline{P}$$

$$\begin{bmatrix} I & 0 \\ -\lambda n^T & \lambda \end{bmatrix}\overline{P} = \begin{pmatrix} x \\ y \\ 1 \\ \mu \end{pmatrix} \equiv P$$

Choice of Canonical Frame

$$p' \cong [H;e'] \begin{pmatrix} p \\ \mu' \end{pmatrix} = [H;e'] \begin{bmatrix} I & 0 \\ n^T & 1/\lambda \end{bmatrix} \begin{pmatrix} p \\ \mu \end{pmatrix}$$

$$[H;e'] \begin{bmatrix} I & 0 \\ n^T & 1/\lambda \end{bmatrix} = \left[H + e'n^T; \quad (1/\lambda)e' \right]$$

$$\cong \left[\lambda H + e'n^T; \quad e' \right]$$

$$\boxed{p' \cong \left[\lambda H + e'n^T; \quad e' \right] \begin{pmatrix} p \\ \mu \end{pmatrix}}$$

where (n^T, λ) are free variables

Family of Homography Matrices

$$p_i^j \cong \left[\lambda H_j + e_j n^T; \quad e_j\right] P_i$$

$$H_\pi = \lambda H + e' n^T$$

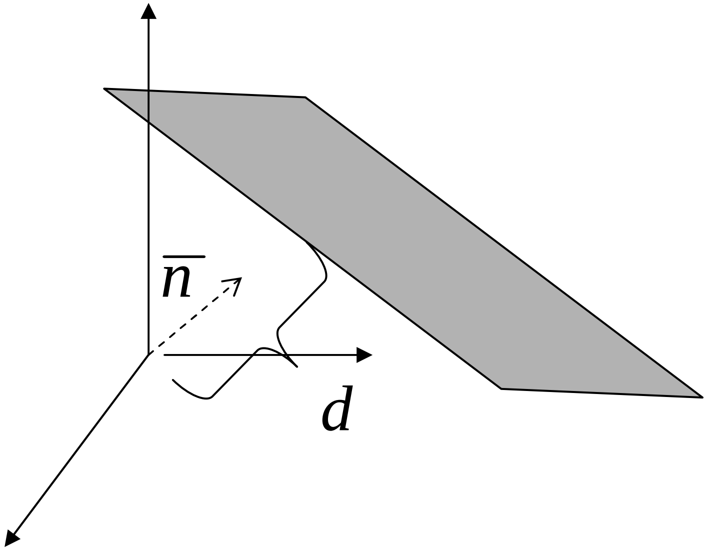
H_π Stands for the family of 2D projective transformations between two fixed images induced by a plane in space

Family of Homography Matrices

$$H_{\pi} \cong H_{\infty} + \frac{1}{d} e' n^{T}$$

when $d \to \infty$

$$H_{\pi} \to H_{\infty} \cong K' R K^{-1}$$



first camera frame

⇨ $K' R K^{-1}$ is the homography matrix induced by the plane at infinity

Reconstruction Problem

$$\boxed{p' \cong [H; \quad e'] \begin{pmatrix} p_i \\ \mu_i \end{pmatrix} = Hp_i + \mu_i e'}$$

We wish to solve for the motion $[H \; ; \; e']$ and structure μ_i from matches $p_i \leftrightarrow p_i'$
Without additional information we cannot solve uniquely for H because
H is determined up to a 4-parameter family (position of a reference plane
in space).

A primer on projective geometry of the plane

Projective Geometry of the Plane

$ax + by + c = 0$ Equation of a line in the 2D plane

⇒ The line is represented by the vector $l = (a, b, c)^T$

and $p^T l = 0$ $p = (x, y, 1)^T$

Correspondence between lines and vectors are not 1-1 because

$(\lambda a, \lambda b, \lambda c)^T$ represents the same line $p^T (\lambda l) = 0, \forall \lambda \neq 0$

The vector $(0,0,0)^T$ does not represent any line.

⇒ Two vectors differing by a scale factor are equivalent. This equivalence class is called homogenous vector. Any vector $(a, b, c)^T$ is a representation of the equivalence class.

Projective Geometry of the Plane

A point (x, y) lies on the line (coincident with) which is represented by $l = (a, b, c)^T$ iff $p^T l = 0$

But also $(\lambda p)^T l = 0$

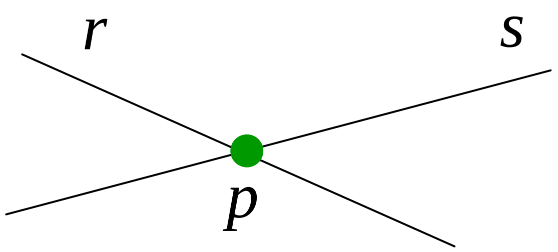
➡ $(\lambda x, \lambda y, \lambda)^T, \forall \lambda \neq 0$ represents the point (x, y)

➡ $(x_1, x_2, x_3)^T$ represents the point $(\frac{x_1}{x_3}, \frac{x_2}{x_3})$

The vector $(0,0,0)^T$ does not represent any point.

Points and lines are dual to each other (only in the 2D case!).

Projective Geometry of the Plane



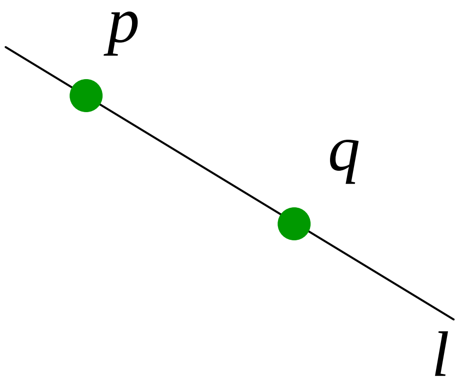
$$p \cong r \times s$$

$$p^T r = (r \times s)^T r = 0$$

$$p^T s = (r \times s)^T s = 0$$

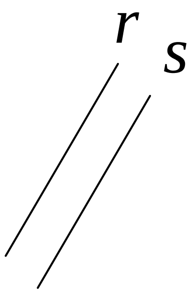
note: $(a \times b)^T c = \det(a, b, c)$

Likewise,



$$l \cong p \times q$$

Lines and Points at Infinity

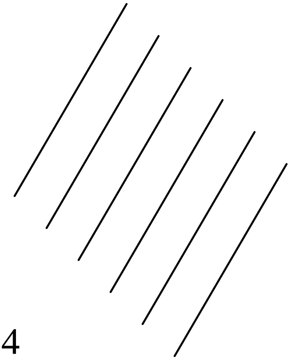


Consider lines $r = (a, b, c)^T \quad s = (a, b, c')^T$

$$r \times s = \begin{pmatrix} bc' - cb \\ ac - ac' \\ 0 \end{pmatrix} = (c' - c) \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} \cong \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$$

point at infinity

which represents the point $(\frac{b}{0}, \frac{-a}{0})$ with infinitely large coordinates



$\begin{pmatrix} a \\ b \\ \lambda \end{pmatrix}$ All meet at the same point $\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$

Lines and Points at Infinity

The points $(x_1, x_2, 0)^T, \forall x_1, x_2$ lie on a line

$$(x_1, x_2, 0)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

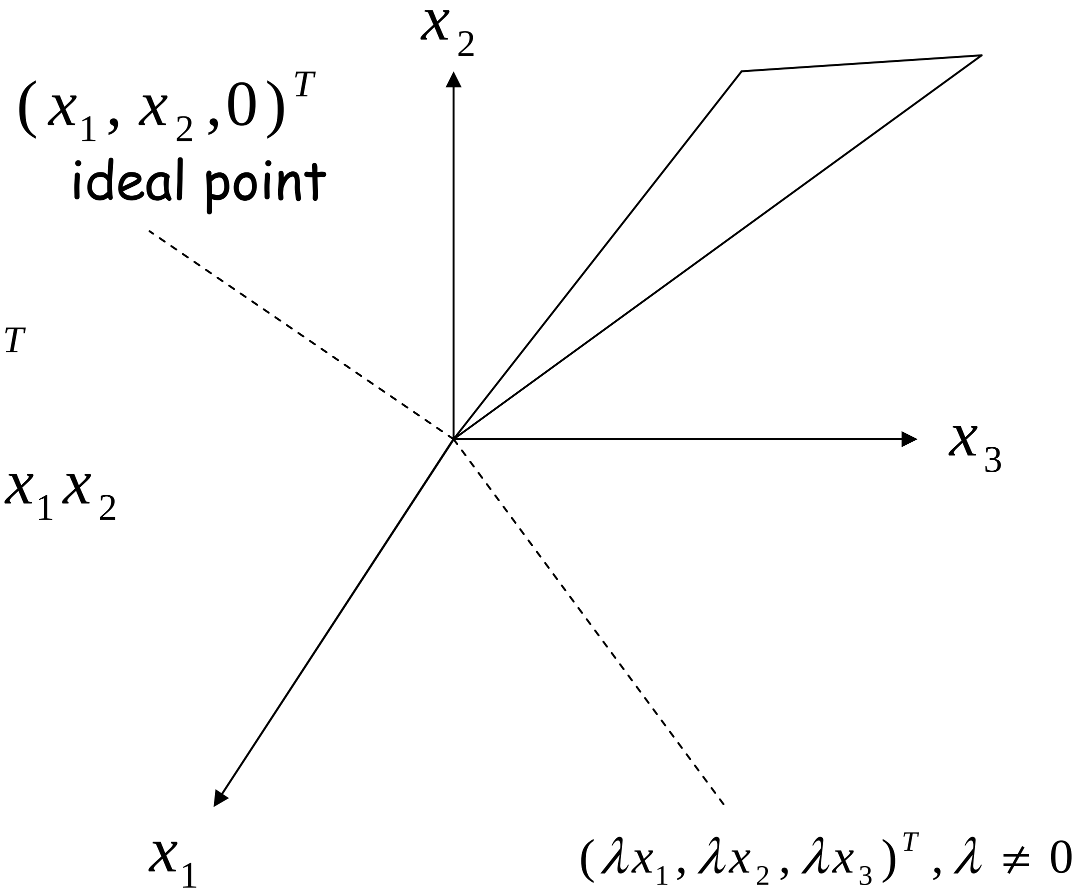
The line $l_\infty = (0,0,1)^T$ is called the line at infinity

The points $(x_1, x_2, 0)^T, \forall x_1, x_2$ are called ideal points.

A line $(a, b, \lambda)^T$ meets $l_\infty = (0,0,1)^T$ at $\begin{pmatrix} a \\ b \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$

(which is the direction of the line)

A Model of the Projective Plane



$l_\infty = (0,0,1)^T$

is the plane $x_1 x_2$

Points are represented as lines (rays) through the origin

Lines are represented as planes through the origin

A Model of the Projective Plane

$$\mathbf{P}^n = \{[x_1,\ldots, x_n] \neq [0,\ldots, 0] : [x_1,\ldots, x_n] = [\lambda x_1,\ldots, \lambda x_n], \forall \lambda \neq 0\}$$

={lines through the origin in R^{n+1} }

={1-dim subspaces of R^{n+1} }

Projective Transformations in P^2

The study of properties of the projective plane that are invariant under a group of transformations.

Projectivity: $h : P^2 \rightarrow P^2$ that maps lines to lines (i.e. preserves colinearity)

Any invertible 3x3 matrix is a Projectivity:

Let p_1, p_2, p_3 Colinear points, i.e.

$$l^T p_i = 0 \Rightarrow l^T H^{-1} H p_i = 0$$

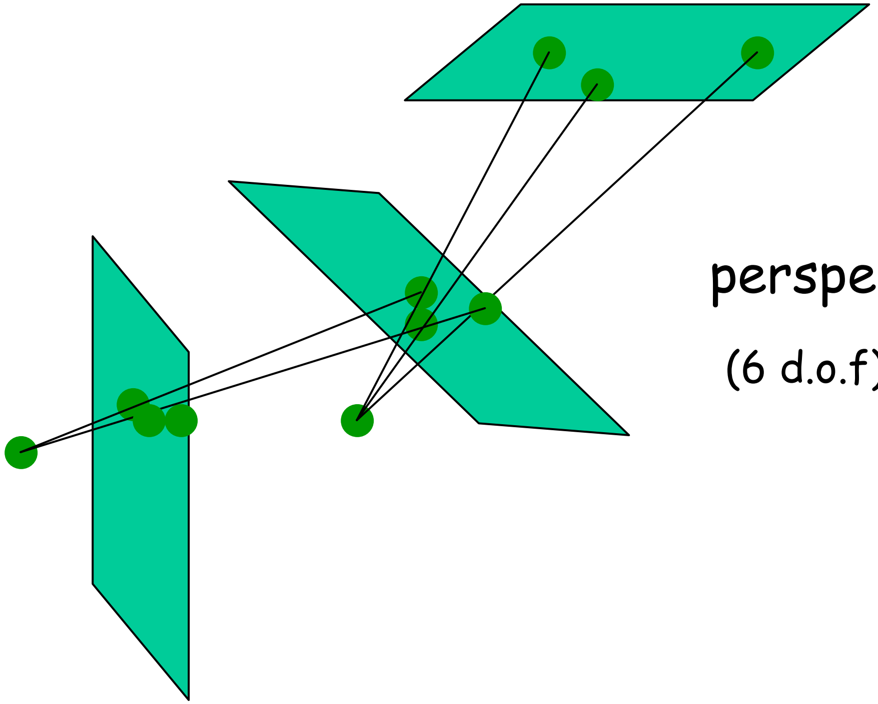
$\Rightarrow$ the points Hp_i lie on the line $H^{-T} l$

Therefore H preserves colinearity.

H is called homography, colineation H^{-T} is the dual.

A homography is determined by 8 parameters.

Projective Transformations in P^2



perspectivity

(6 d.o.f)

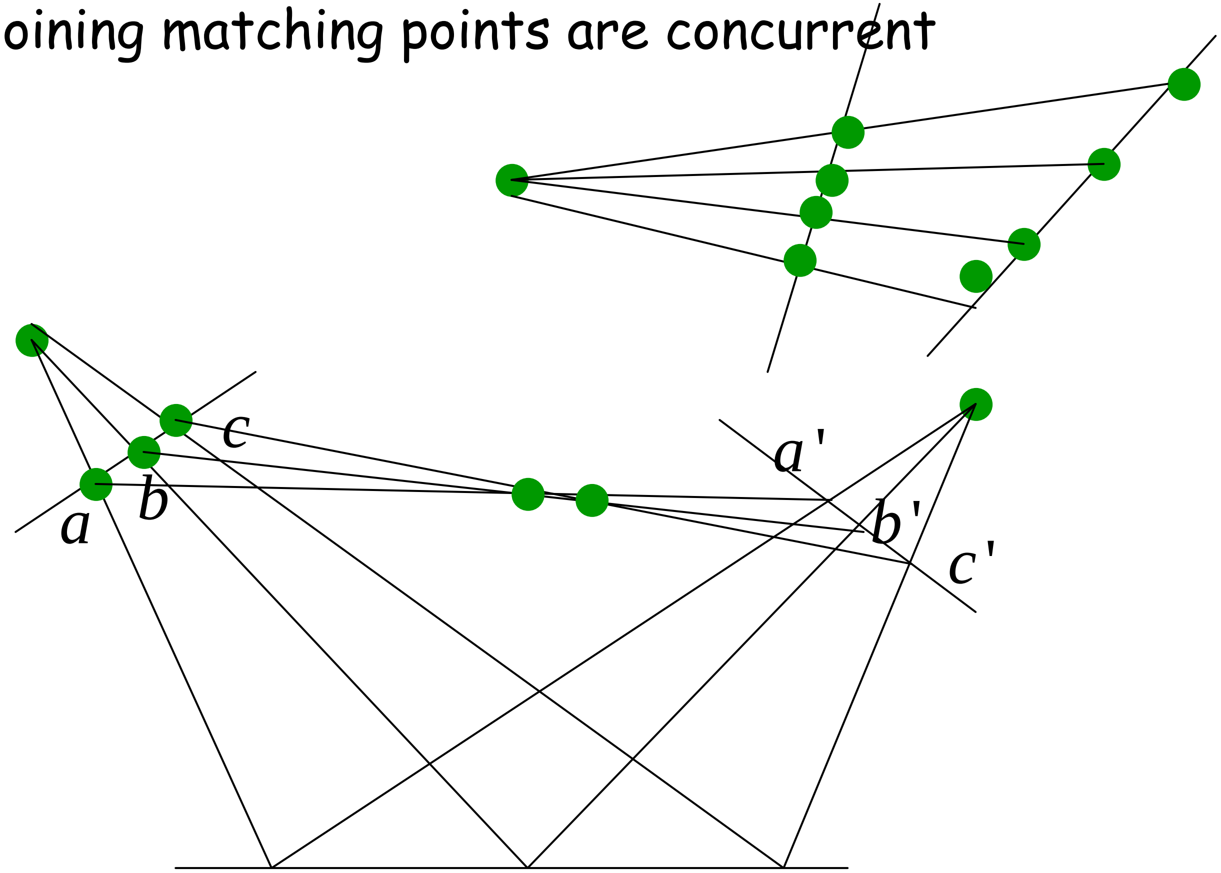
A composition of perspectivities from a plane π to other planes and back to π is a projectivity.

Every projectivity can be represented in this way.

Projective Transformations in P^2

Example, a prespectivity in 1D:

Lines adjoining matching points are concurrent



Lines adjoining matching points (a,a'),(b,b'),(c,c') are not concurrent

Projective Transformations in P^2

$l_\infty = (0,0,1)^T$ is not invariant under H:

Points on l_∞ are $(x_1, x_2, 0)^T$

$$H\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 h_1 + x_2 h_2 = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

x'_3 is not necessarily 0

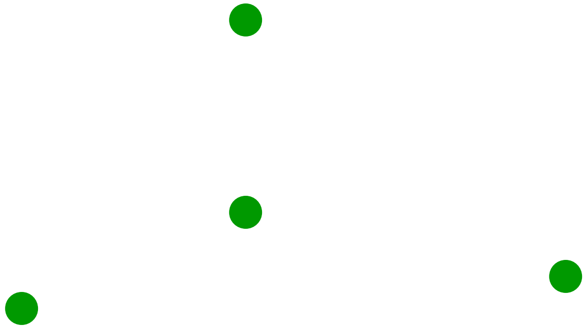
⇒ Parallel lines do not remain parallel !

⇒ l_∞ is mapped to $H^{-T} l_\infty$

Projective Basis

A Simplex in R^{n+1} is a set of n+2 points such that no subset Of n+1 of them lie on a hyperplane (linearly dependent).

In P^2 a Simplex is 4 points



Theorem: there is a unique colineation between any two Simplexes

Projective Invariants

Invariants are measurements that remain fixed under colineations

# of independent invariants = # d.o.f of configuration - # d.o.f of trans.

Ex: 1D case $p' \cong H_{2x2}\, p$ → H has 3 d.o.f

A point in 1D is represented by 1 parameter.

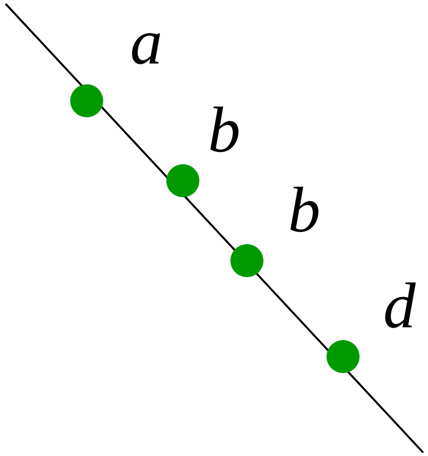
4 points we have: 4-3=1 invariant (cross ratio)

2D case: H has 8 d.o.f, a point has 2 d.o.f thus 5 points induce 2 invariants

Projective Invariants

The cross-ratio of 4 points:

$$\alpha = \frac{\frac{\overline{ab}}{\overline{ac}}}{\frac{\overline{cd}}{\overline{bd}}}$$



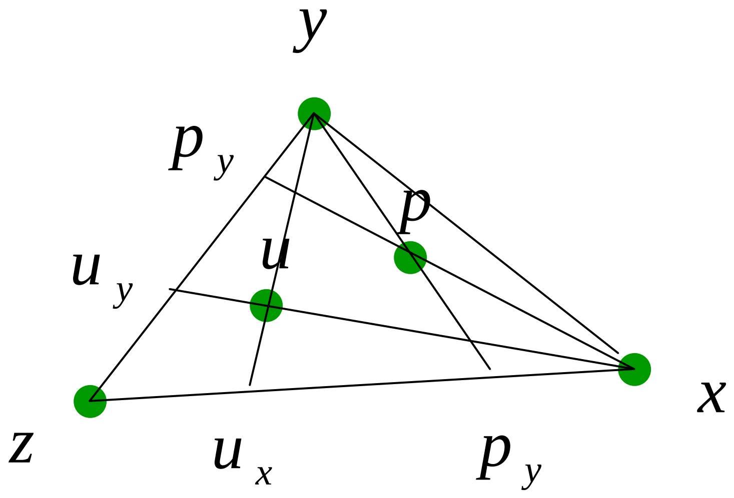
24 permutations of the 4 points forming 6 groups:

$$\alpha, \frac{1}{\alpha}, 1-\alpha, \frac{\alpha-1}{\alpha}, \frac{\alpha}{\alpha-1}, \frac{1}{1-\alpha}$$

Projective Invariants

5 points gives us 10 d.o.f, thus 10-8=2 invariants which represent 2D

x, y, z, u are the 4 basis points (simplex)



$\alpha =< z, u_x, p_x, x >$

$\beta =< z, u_y, p_y, y >$

→ p_x, p_y are determined uniquely by α, β

Point of intersection is preserved under projectivity (exercise)

→ p_x, p_y uniquely determined p

Epipolar Geometry and Fundamental Matrix

Reminder:

$$p \cong [I;0]\overline{P}$$

$$p = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \overline{P} = \begin{pmatrix} p \\ \mu' \end{pmatrix}$$

$$p' \cong [H;e']\overline{P}$$

$$p' \cong \left[\lambda H + e'n^T; \quad e'\right]P \qquad P = \begin{pmatrix} p \\ \mu \end{pmatrix}$$

$$\longrightarrow \quad \boxed{p' \cong H_\pi p + \mu e'}$$

$$H_\pi = \lambda H + e'n^T$$

H_π Stands for the family of 2D projective transformations between two fixed images induced by a plane in space

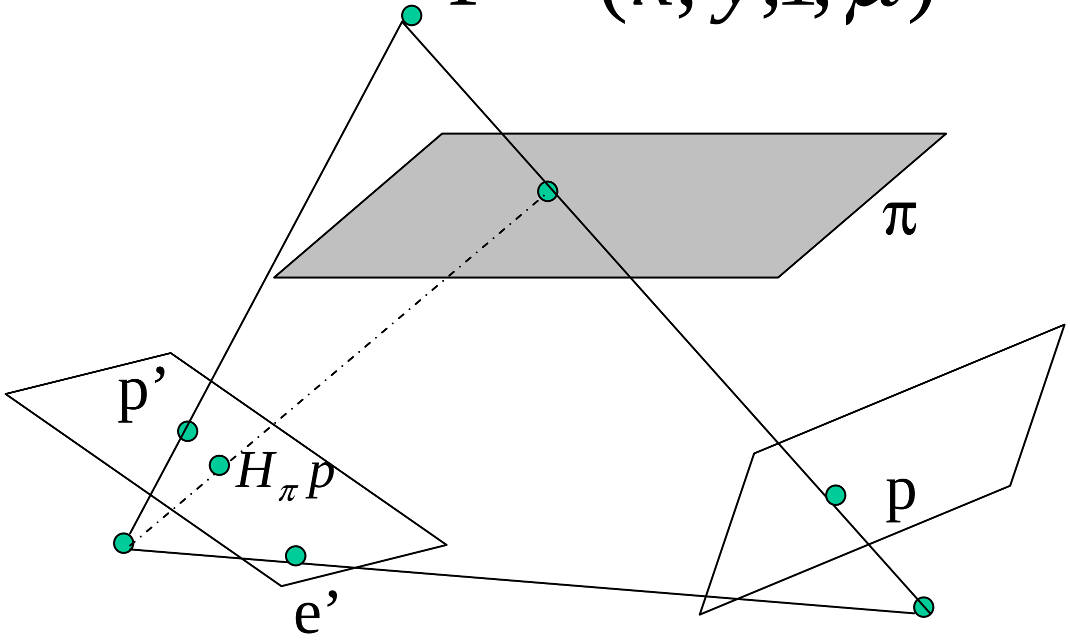
Plane + Parallax

$$\boxed{p' \cong H_\pi p + \mu e'}$$

$$p \cong [I,0]P_\pi$$

$$p' \cong [H_\pi \quad e']P$$

$$P = (x, y, 1, \mu)$$



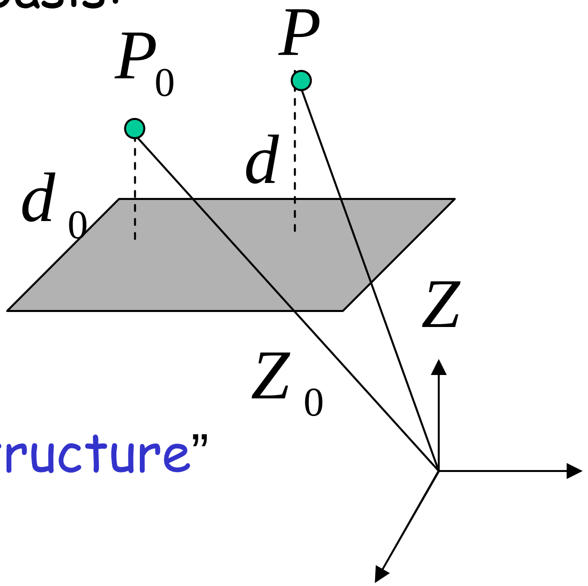
- what does μ stand for?
- what would we obtain after eliminating μ

Plane + Parallax

$$\boxed{p' \cong H_{\pi} p + \mu e'}$$

$$\mu = \frac{Z_0}{Z} \frac{d}{d_0}$$

We have used 4 space points for a basis:
3 for the reference plane
1 for the reference point (scaling)



➡ Since 4 points determine an affine basis:

μ is called "relative affine structure"

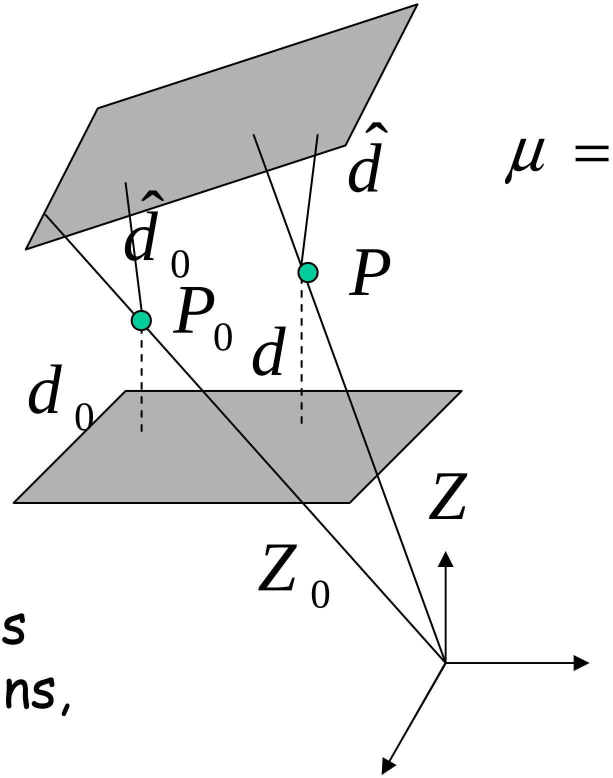
Note: we need 5 points for a projective basis. The 5th point is the first camera center.

Note: A projective invariant

$$p' \cong H_{\pi} p + \mu e'$$

$$p' \cong H_{\hat{\pi}} p + \hat{\mu} e'$$

$$\frac{\mu}{\hat{\mu}} = \frac{\hat{d}_0}{\hat{d}} \frac{d}{d_0}$$



$$\mu = \frac{Z_0}{Z} \frac{d}{d_0}$$

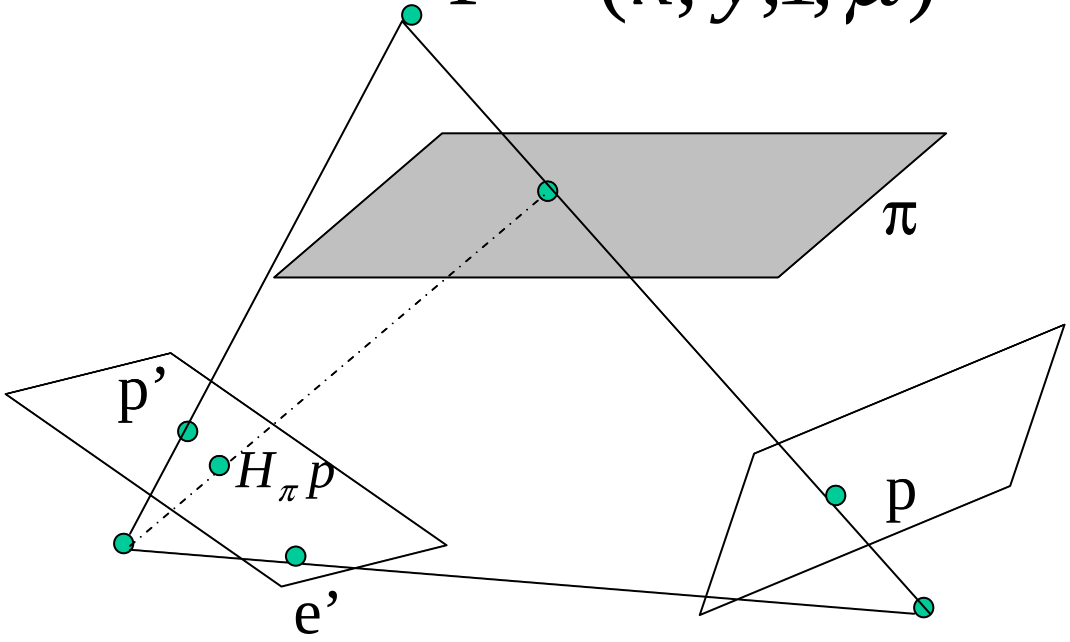
This invariant ("projective depth") is independent of both camera positions, therefore is projective.

5 basis points: 4 non-coplanar defines two planes, and A 5th point for scaling.

Fundamental Matrix

$$\boxed{p' \cong H_\pi p + \mu e'}$$

$$P = (x, y, 1, \mu)$$



$$rank\begin{bmatrix} p' & H_\pi p & e' \end{bmatrix} = 2$$

$$\Longrightarrow \quad p'^T (e' \times H_\pi p) = 0$$

$$p'^T ([e']_\times H_\pi) p = 0 \qquad\qquad p'^T F p = 0$$

Fundamental Matrix

$$p'^T ([e']_\times H_\pi) p = 0$$

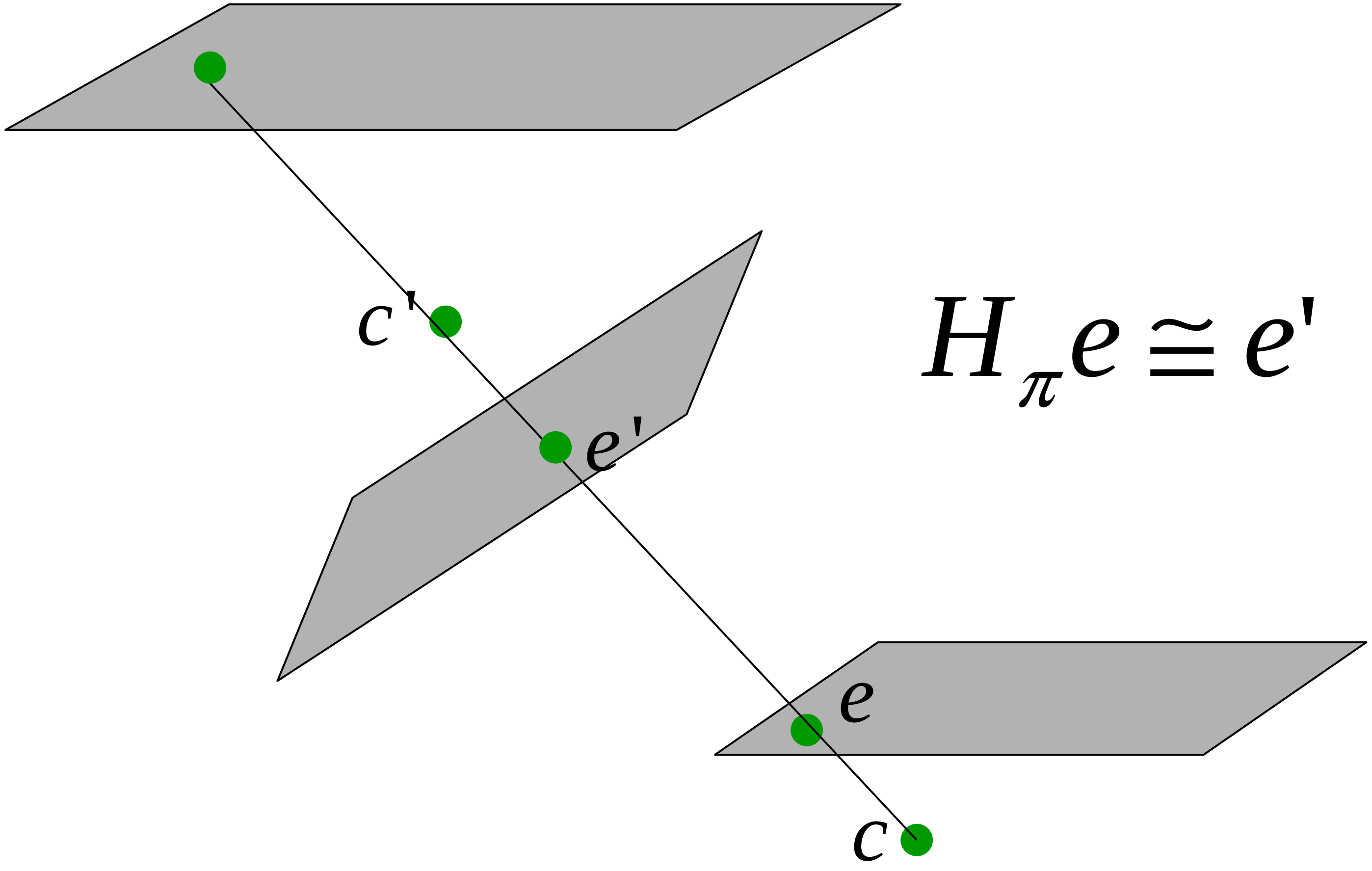
$p'^T Fp = 0$ Defines a bilinear matching constraint whose coefficients depend only on the camera geometry (shape was eliminated)

- F does not depend on the choice of the reference plane

$$[e']_\times H_\pi = [e']_\times (\lambda H_\infty + e' n^T) \cong [e']_\times H_\infty$$

Epipoles from F

Note: any homography matrix maps between epipoles:



$$H_\pi e \cong e'$$

Epipoles from F

$$Fe = 0$$

$$[e']_\times H_\infty e \cong [e']_\times e' = 0$$

$$F^T e' = 0$$

$$-H_\infty^{\ T} [e']_\times e' = 0$$

Fp is the epipolar line of p - the projection of the line of sight onto the second image.

Estimating F from matching points

$p_i'^T F p_i = 0$ $\qquad i = 1,\dots,8$ $\qquad$ Linear solution

$p_i'^T F p_i = 0$ $\qquad i = 1,\dots,7$ $\qquad$ N on-linear solution

$\det(F) = 0$

$\det(F) = 0$ is cubic in the elements of F, thus we should expect 3 solutions.

Estimating F from Homographies

$H_\pi^T F$ is skew-symmetric (i.e. provides 6 constraints on F)

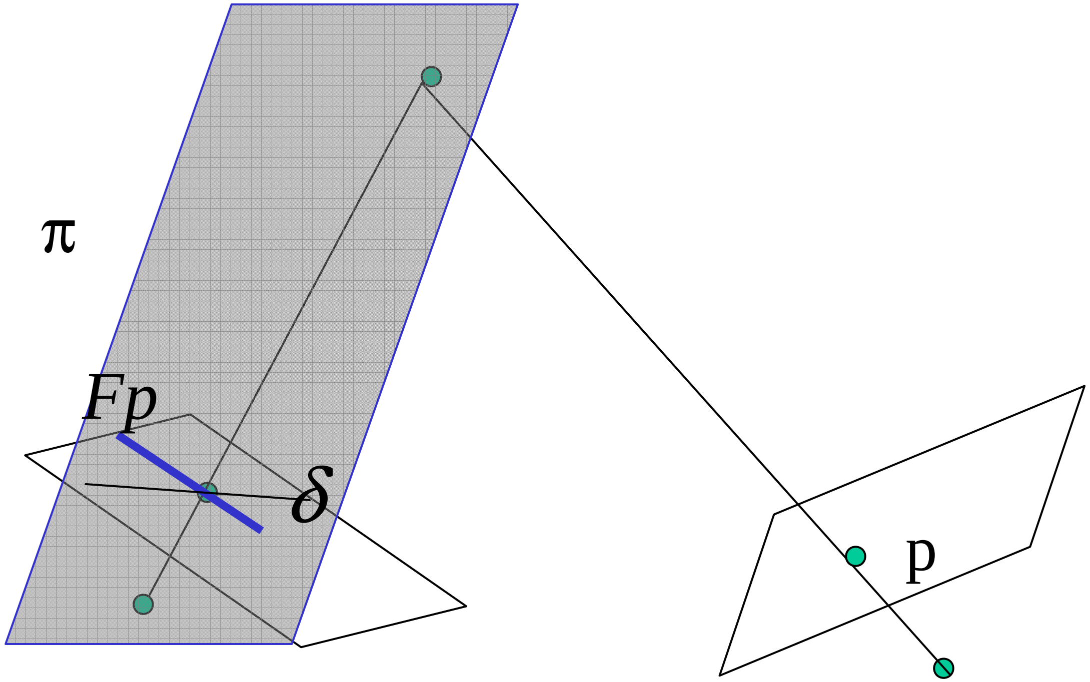
$$H_\pi^T F = (\lambda H_\infty + e' n^T)^T [e']_\times H_\infty = \lambda H_\infty^T [e']_\times H_\infty$$

$$F^T H_\pi = -H_\infty^T [e']_\times (\lambda H_\infty + e' n^T) = -\lambda H_\infty^T [e']_\times H_\infty$$

⇒ $H_\pi^T F = -F^T H_\pi$

⇒ 2 homography matrices are required for a solution for F

F Induces a Homography



$[\delta]_\times F$ is a homography matrix induced by the plane defined by the join of the image line δ and the camera center

Projective Reconstruction

1. Solve for F via the system $p_i'^T F p_i = 0$ (8 points or 7 points)

2. Solve for e' via the system $F^T e' = 0$

3. Select an arbitrary vector δ $\delta^T e' \neq 0$

4. $\begin{bmatrix} I & 0 \end{bmatrix}$ and $\begin{bmatrix} [\delta]_\times F & e' \end{bmatrix}$ are a pair of camera matrices.

$$\boxed{p' \cong [\delta]_\times Fp + \mu e'}$$

Why 3 views?

Trifocal Geometry

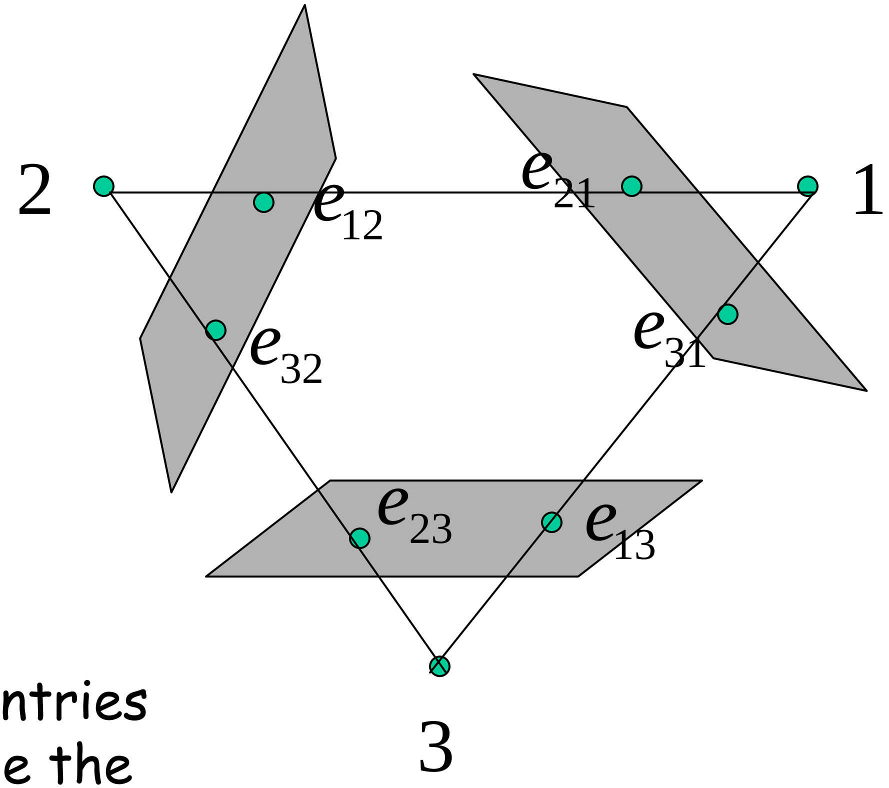
The three fundamental matrices completely describe the trifocal geometry (as long as the three camera centers are not collinear)

$$F_{12}e_{31} = e_{12} \times e_{32}$$

➡ $$e_{32}^{T}F_{12}e_{31} = 0$$

Likewise: $$e_{13}^{T}F_{23}e_{12} = 0$$

$$e_{23}^{T}F_{13}e_{21} = 0$$



Each constraint is non-linear in the entries of the fundamental matrices (because the epipoles are the respective null spaces)

Trifocal Geometry

$$e_{32}^T F_{12} e_{31} = 0$$

$$e_{13}^T F_{23} e_{12} = 0$$

$$e_{23}^T F_{13} e_{21} = 0$$

3 fundamental matrices provide 21 parameters. Subtract 3 constraints,
Thus we have that the trifocal geometry is determined by 18 parameters.

This is consistent with the straight-forward counting:

3x11 – 15 = 18

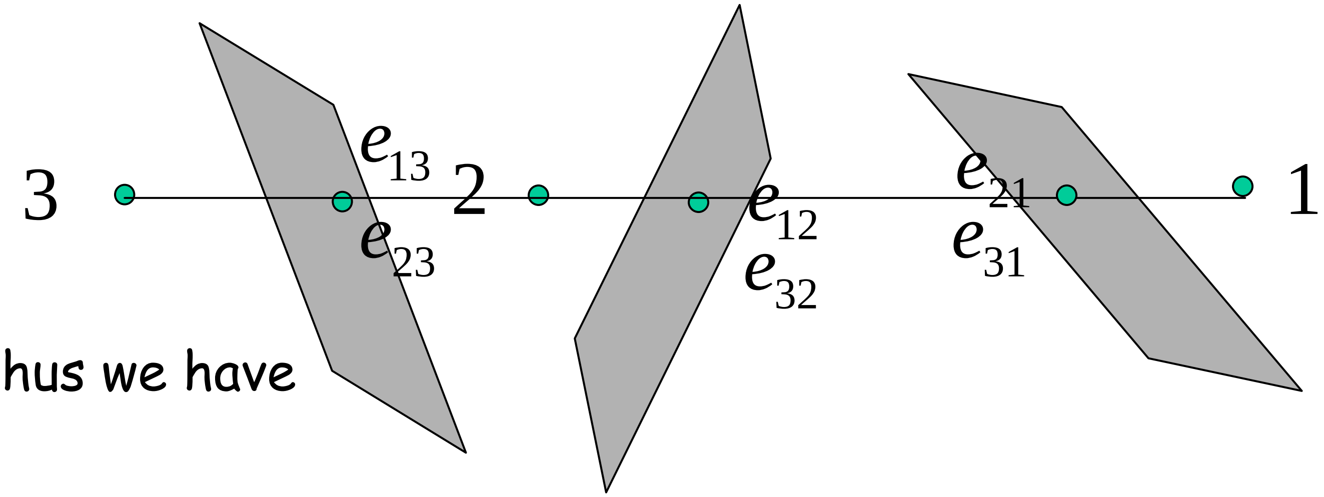
(3 camera matrices provide 33 parameters, minus the projective basis)

What Goes Wrong with 3 views?

$e_{31} \cong e_{21}$

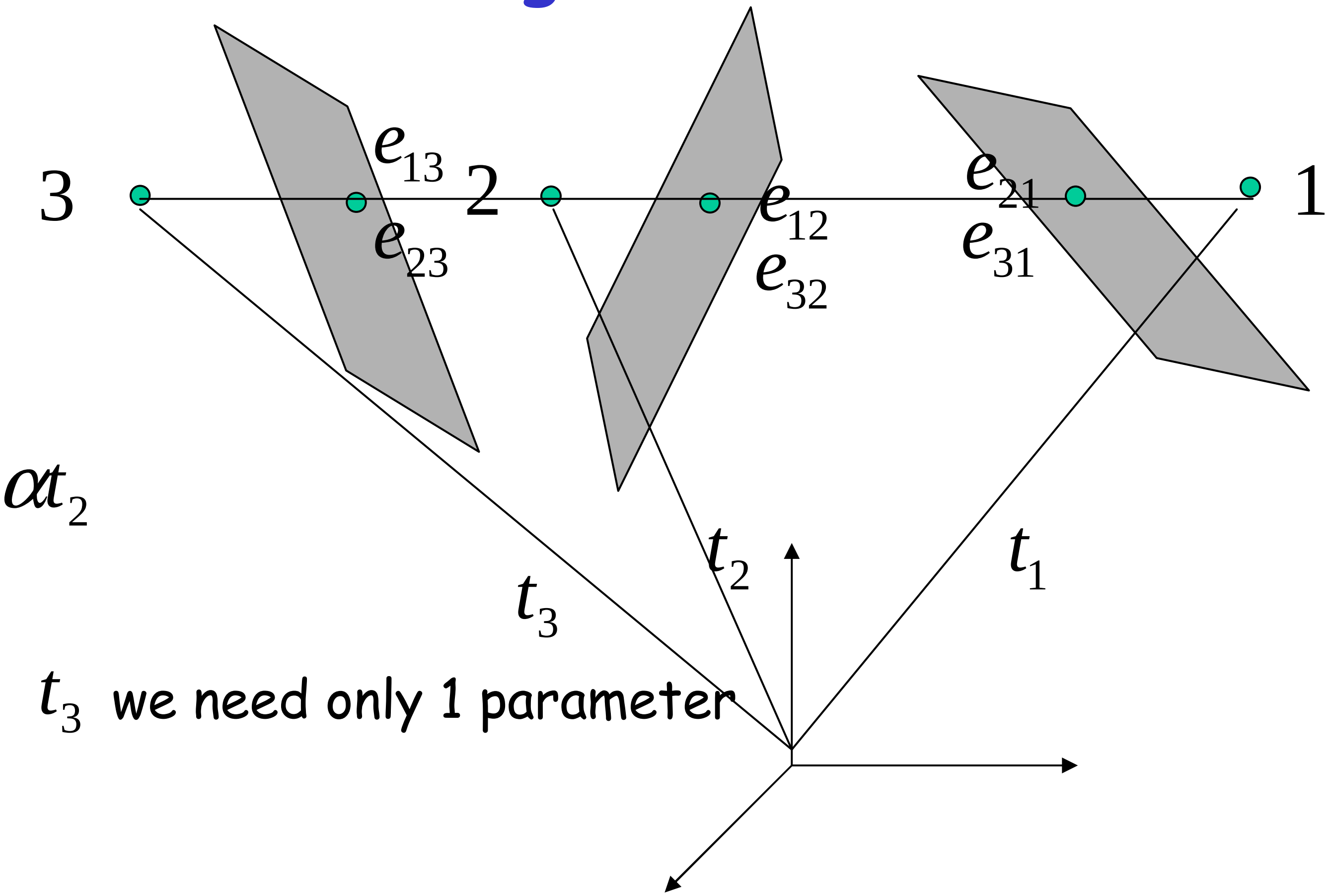
$e_{12} \cong e_{32}$

$e_{13} \cong e_{23}$



2 constraints each, thus we have
21-6=15 parameters

What Goes Wrong with 3 views?



$$t_3 = t_1 + \alpha t_2$$

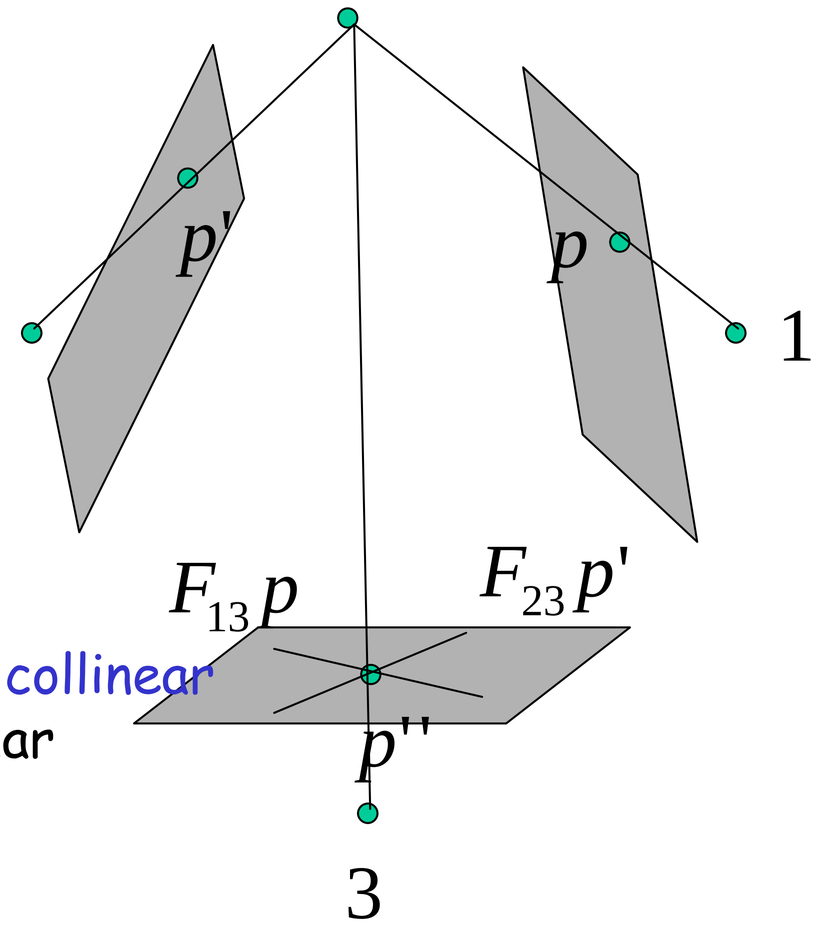
Thus, to represent t_3 we need only 1 parameter (instead of 3).

18-2=16 parameters are needed to represent the trifocal geometry in this case.

but the pairwise fundamental matrices can account for only 15!

What Else Goes Wrong: Reprojection

$$p'' \cong F_{13}p \times F_{23}p'$$



Given p,p’ and the pairwise F-mats one can directly determine the position of the matching point p’’

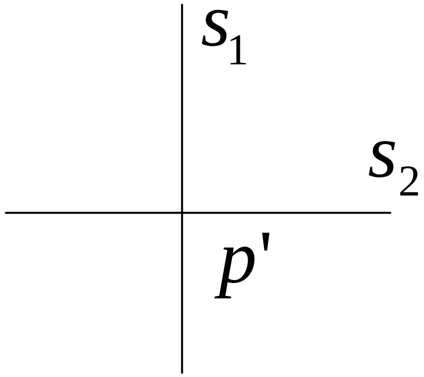
This fails when the 3 camera centers are collinear because all three line of sights are coplanar thus there is only one epipolar line!

Trifocal Constraints

The Trifocal Constraints

$$p = [I \quad 0]P \qquad p' \cong [A \quad e']P \qquad p'' \cong [B \quad e'']P$$

$$s_1 = \begin{pmatrix} -1 \\ 0 \\ x' \end{pmatrix} \quad s_2 = \begin{pmatrix} 0 \\ -1 \\ y' \end{pmatrix} \Longrightarrow \begin{array}{l} s_1^T p' = 0 \\ s_2^T p' = 0 \end{array}$$



$$r_1 = \begin{pmatrix} -1 \\ 0 \\ x'' \end{pmatrix} \quad r_2 = \begin{pmatrix} 0 \\ -1 \\ y'' \end{pmatrix} \Longrightarrow \begin{array}{l} r_1^T p'' = 0 \\ r_2^T p'' = 0 \end{array}$$

The Trifocal Constraints

$$s_1^T p' = 0$$

$$s_2^T p' = 0$$

$$p' \cong [A \quad e']P \Longrightarrow \begin{array}{l} s_1^T [A \quad e']P = 0 \\ s_2^T [A \quad e']P = 0 \end{array}$$

$$r_1^T p'' = 0$$

$$r_2^T p'' = 0$$

$$p'' \cong [B \quad e'']P \Longrightarrow \begin{array}{l} r_1^T [B \quad e'']P = 0 \\ r_2^T [B \quad e'']P = 0 \end{array}$$

$$p = [I \quad 0]P \Longrightarrow \begin{array}{l} \begin{pmatrix} -1 & 0 & x & 0 \end{pmatrix} P = 0 \\ \begin{pmatrix} 0 & -1 & y & 0 \end{pmatrix} P = 0 \end{array}$$

The Trifocal Constraints

$$\begin{bmatrix} -1 & 0 & x & 0 \\ 0 & -1 & y & 0 \\ & s_1^T A & & s_1^T e' \\ & s_2^T A & & s_2^T e' \\ & r_1^T B & & r_1^T e'' \\ & r_2^T B & & r_2^T e'' \end{bmatrix}_{6\times 4} P = 0$$

Every 4x4 minor must vanish!

12 of those involve all 3 views, they are arranged in 3 groups
Depending on which view is the reference view.

The Trifocal Constraints

$$\begin{bmatrix} -1 & 0 & x & 0 \\ 0 & -1 & y & 0 \\ & s_1^T A & & s_1^T e' \\ & s_2^T A & & s_2^T e' \\ & r_1^T B & & r_1^T e'' \\ & r_2^T B & & r_2^T e'' \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{The reference view} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Choose 1 row from here} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Choose 1 row from here} \end{array}$$

We should expect to have 4 matching constraints $f_i(p, p', p'') = 0$

The Trifocal Constraints

Expanding the determinants:

$$p' \cong Ap + \mu e' \quad \Longrightarrow \quad s_i^T Ap + \mu s_i^T e' = 0 \qquad i = 1,2$$

$$p'' \cong Bp + \mu e'' \quad \Longrightarrow \quad r_j^T Bp + \mu r_j^T e'' = 0 \qquad j = 1,2$$

eliminate μ

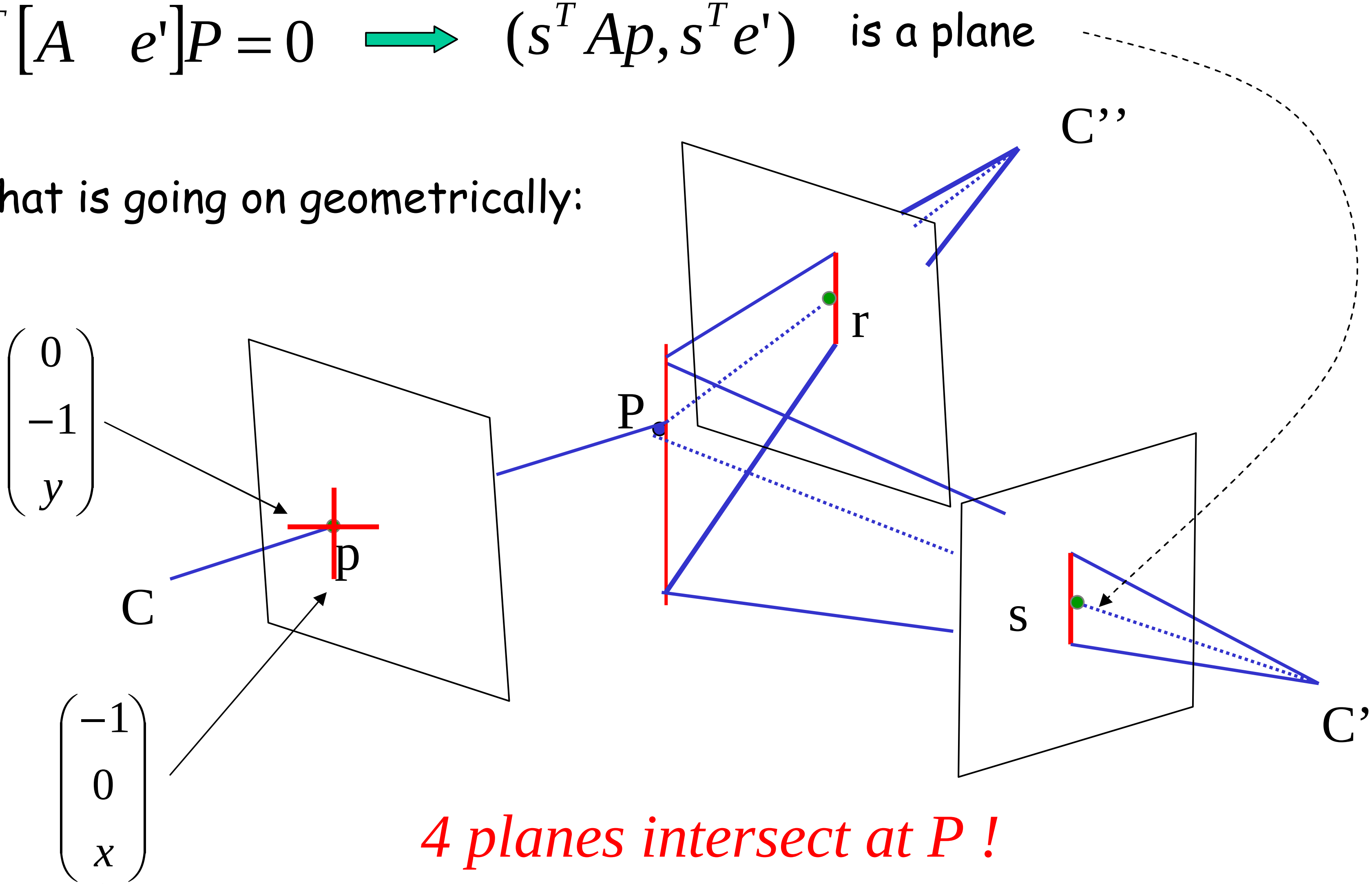
$$\frac{s_i^T Ap}{s_i^T e'} = \frac{r_j^T Bp}{r_j^T e''}$$

$$\Longrightarrow \quad \boxed{(r_j^T e'')(s_i^T Ap) = (s_i^T e')(r_j^T Bp)} \quad i, j = 1,2$$

The Trifocal Constraints

$$s^T[A \quad e']P = 0 \quad \Rightarrow \quad (s^T Ap, s^T e')$$ is a plane

What is going on geometrically:



4 planes intersect at P !

Primer on Covariant-Contravariant conventions

Index Notations

Goal: represent the operations of inner-product and outer-product

A vector has super-script running index when it represents a *point*

A vector has subscript running index when it represents a *hyperplane*

Example:

$$p^i = (p^1, p^2, p^3)$$

Represents a *point* in the projective plane

$$s_j = (s_1, s_2, s_3)$$

Represents a *line* in the projective plane

An *outer-product*:

$u_i v_j$ is an object (2-valence tensor) whose entries are

$$u_1 v_1, \dots, u_1 v_m, \dots, u_n v_1, \dots, u_n v_m$$

Note: this is the outer-product of two vectors:

$$uv^T = \begin{bmatrix} u_1 v_1 & \dots & u_1 v_m \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_n v_1 & \dots & u_n v_m \end{bmatrix}_{n \times m}$$

(rank-1 matrix)

$$a_{ij} = c_i c'_j + d_i d'_j + \dots + x_i x'_j$$

A general 2-valence tensor is a sum of rank-1 2-valence tensors

Likewise,

$$u_i v^j$$

$$u^i v^j$$

$$u^i v_j$$

Are outer-products consisting of the same elements, but as a mapping carry each a different meaning (described later).

These are also called *mixed* tensors, where the super-script is called *contra-variant* index and the *subscript* is called covariant index.

The *inner-product* (contraction):

Summation rule: same index in contravariant and covariant positions are summed over. This is sometimes called the "Einstein summation convention".

$$u_j v^j = u_1 v^1 + u_2 v^2 + \ldots + u_n v^n$$

The *inner-product* (contraction):

$$a_i{}^j u^i = a_1{}^j u^1 + a_2{}^j u^2 + \ldots + a_n{}^j u^n = v^j$$

Note: this is the familiar matrix-vectors multiplication: $Au = v$
where the super-script j runs over the rows of the matrix

Note: the 2-valence tensor a_i^j maps points to points

Likewise,

$a_i^{\ j} u_j = v_i$ Maps hyperplanes (lines in 2D) to hyperplanes

Note: this is equivalent to $A^T u = v$

We have seen in the past that if $Hp \cong p'$ is a homography

Then $H^{-T} l \cong l'$ maps lines from view 1 to view 2

Let p_1, p_2, p_3 Colinear points, i.e.

$l^T p_i = 0 \Longrightarrow l^T H^{-1} H p_i = 0$

$\Longrightarrow$ the points Hp_i lie on the line $H^{-T} l$

With the index notations we get this property immediately!

The complete list:

$a_i{}^j u^i = v^j$ Maps points to points

$a_i{}^j u_j = v_i$ Maps hyperplanes (lines in 2D) to hyperplanes

$a_{ij} u^i = v_j$ Maps points to hyperplanes

$a^{ij} u_i = v^j$ Maps hyperplanes to points

More Examples:

runs over the rows

$$a^k_j b^j_i = c^k_i$$ This is the matrix product $AB = C$

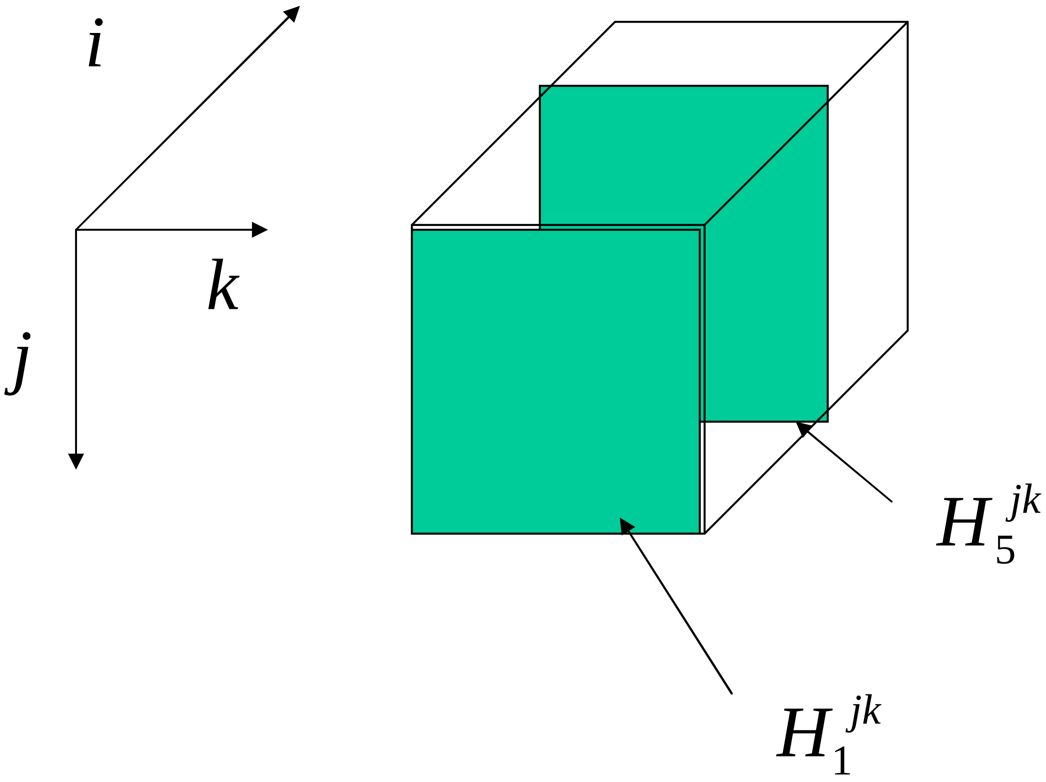
runs over the columns

$$u^i v_j H^{jk}_i$$ Must be a point

➡ H^{jk}_i Takes a point in first frame, a hyperplane in the second frame and produces a point in the third frame

$$u^i H^{jk}_i$$ Must be a matrix (2-valence tensor)

if u=(1,0,0…0) then this is a *slice* H^{jk}_1 of the tensor.

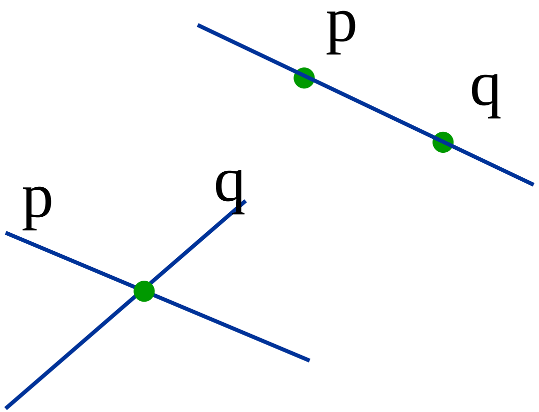

i
k
j
H_5^{jk}
H_1^{jk}

The Cross-product Tensor

$$s = p \times q = \begin{bmatrix} \det\begin{bmatrix} p_2 & q_2 \\ p_3 & q_3 \end{bmatrix} \\ -\det\begin{bmatrix} p_1 & q_1 \\ p_3 & q_3 \end{bmatrix} \\ \det\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} \end{bmatrix}$$

$$\varepsilon_{ijk} p^i q^j = s_k$$

$$\varepsilon^{ijk} p_i q_j = s^k$$



$$u \times v = [u]_\times v$$

$$[u]_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

The cross-product tensor is defined such that $\varepsilon^{ijk} u_i$

Produces the matrix $[u]_\times$ i.e., the entries are 1,-1,0

$$\varepsilon^{1jk} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \varepsilon^{2jk} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \varepsilon^{3jk} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$u_i \varepsilon^{ijk} = u_1 \varepsilon^{1jk} + u_2 \varepsilon^{2jk} + u_3 \varepsilon^{3jk} = [u]_\times$$

$$[u]_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

The Trifocal Tensor

The Trifocal Tensor

$$\boxed{(r_j^T e'')(s_i^T Ap) = (s_i^T e')(r_j^T Bp)}$$

New index notations: i-image 1, j-image 2, k-image 3

$$s^T Ap + \mu s^T e' = 0 \quad \Longrightarrow \quad s_j a_i^j p^i + \mu s_j e'^j = 0$$

p^i is a point in image 1

s_j is a line in image 2

e'^j is a point in image 2

The Trifocal Tensor

s_j^l $l = 1,2$ are the two lines coincident with p', i.e. $s_j^l p'^j = 0$

r_k^m $m = 1,2$ are the two lines coincident with p'', i.e. $r_k^m p''^k = 0$

$$s_j^l a_i^j p^i + \mu s_j^l e'^j = 0$$

$$r_k^m b_i^k p^i + \mu r_k^m e''^k = 0$$

Eliminate μ

$$(s_j^l e'^j)(r_k^m b_i^k p^i) - (r_k^m e''^k)(s_j^l a_i^j p^i) = 0$$

The Trifocal Tensor

$$(s_j^l e'^j)(r_k^m b_i^k p^i) - (r_k^m e''^k)(s_j^l a_i^j p^i) = 0$$

Rearrange terms:

$$p^i s_j^l r_k^m (e'^j b_i^k - e''^k a_i^j) = 0 \qquad l, m = 1,2$$

The trifocal tensor is:

$$T_i^{jk} = e'^j b_i^k - e''^k a_i^j$$

The Trifocal Tensor

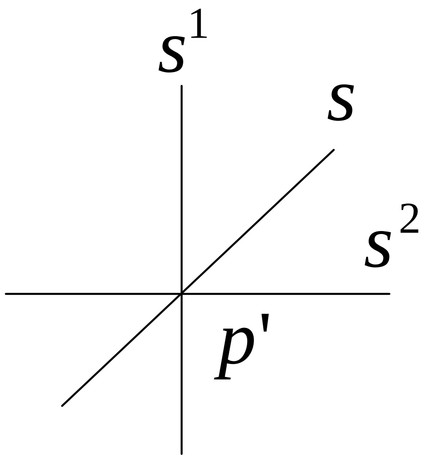
$$p^i s_j^l r_k^m T_i^{jk} = 0$$

$$s^l{}_j = \begin{bmatrix} -1 & 0 & x' \\ 0 & -1 & y' \end{bmatrix} \qquad r^m{}_k = \begin{bmatrix} -1 & 0 & x'' \\ 0 & -1 & y'' \end{bmatrix}$$

The four “trilinearities”:

$$x'' T_i^{13} p^i - x''x' T_i^{33} p^i + x' T_i^{31} p^i - T_i^{11} p^i = 0$$
$$y'' T_i^{13} p^i - y''x' T_i^{33} p^i + x' T_i^{32} p^i - T_i^{12} p^i = 0$$
$$x'' T_i^{23} p^i - x''y' T_i^{33} p^i + y' T_i^{31} p^i - T_i^{21} p^i = 0$$
$$y'' T_i^{23} p^i - y''y' T_i^{33} p^i + x' T_i^{32} p^i - T_i^{22} p^i = 0$$

The Trifocal Tensor



$$s_j = \alpha s_j^1 + \beta s_j^2$$

$$r_k = \gamma r_k^1 + \delta r_k^2$$

$$p^i s_j r_k T_i^{jk} = p^i (\alpha s_j^1 + \beta s_j^2)(\gamma r_k^1 + \delta r_k^2) T_i^{jk} = 0$$

A trilinearity is a contraction with a point-line-line where the lines are coincident with the respective matching points.

Slices of the Trifocal Tensor

Now that we have an explicit form of the tensor, what can we do with it?

$$p^i s_j T_i^{jk} = ?$$

The result must be a contravariant vector (a point). This point is coincident with r for all lines coincident with p''

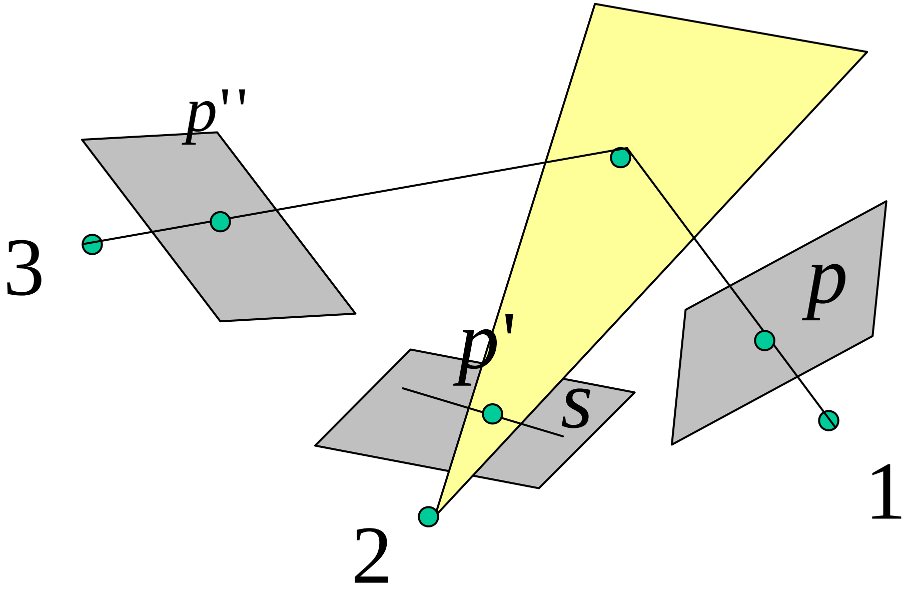
➡ $$p^i s_j T_i^{jk} \cong p''^k \qquad s \neq e' \times p'$$

The point reprojection equation (will work when camera centers are collinear as well).

Note: reprojection is possible after observing 7 matching points, (because one needs 7 matching triplets to solve for the tensor). This is in contrast to reprojection using pairwise fundamental matrices Which requires 8 matching points (in order to solve for the F-mats).

Slices of the Trifocal Tensor

$$\boxed{p^i s_j T_i^{jk} \cong p''^k}$$



Slices of the Trifocal Tensor

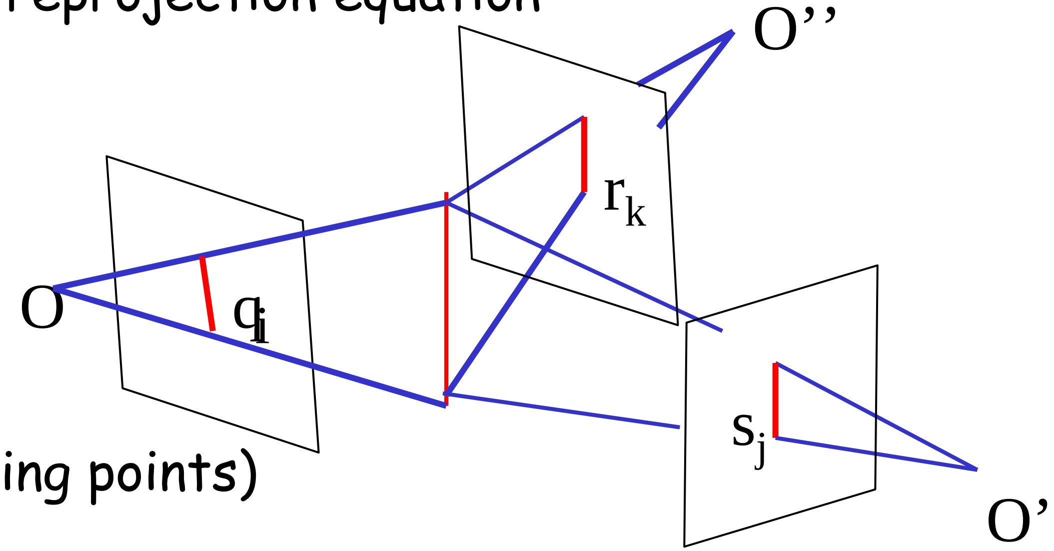
$$s_j r_k T_i^{jk} = ?$$

The result must be a line.

$$s_j r_k T_i^{jk} \cong q_i$$ Line reprojection equation

➡

13 matching lines
are necessary for
solving for the tensor
(compared to 7 matching points)



Slices of the Trifocal Tensor

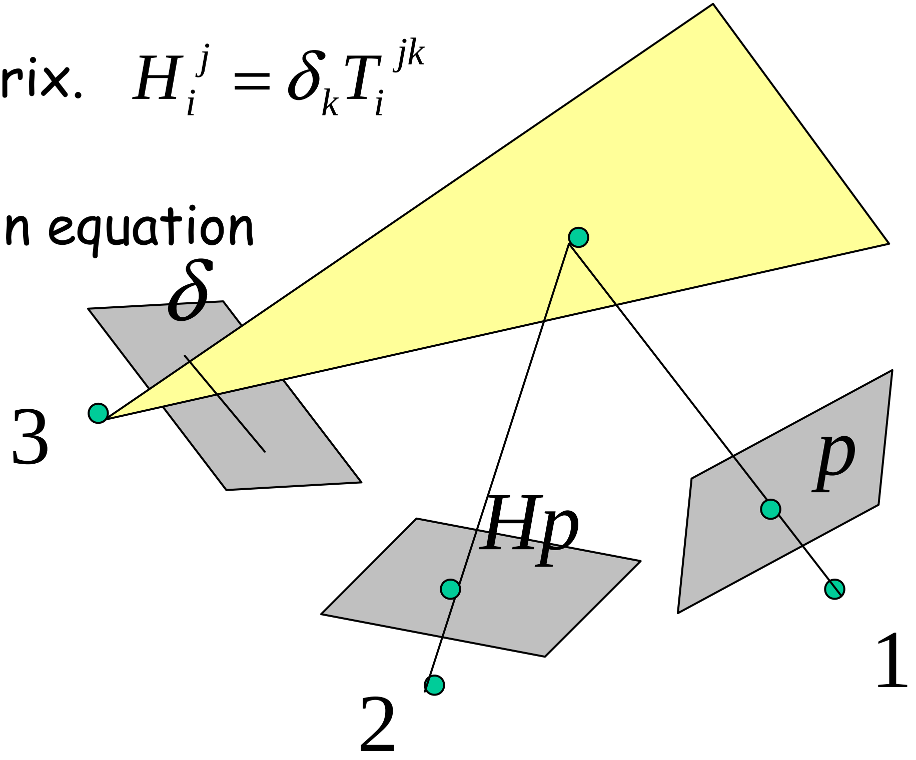
$$\delta_k T_i^{jk} = ?$$

The result must be a matrix. $H_i^j = \delta_k T_i^{jk}$

$p^i \delta_k T_i^{jk}$ is the reprojection equation

➡

H is a homography matrix



$$\delta_k T_i^{jk} \quad \forall \delta$$

is a family of homography matrices (from 1 to 2) induced by the family of planes coincidant with the 3rd camera center.

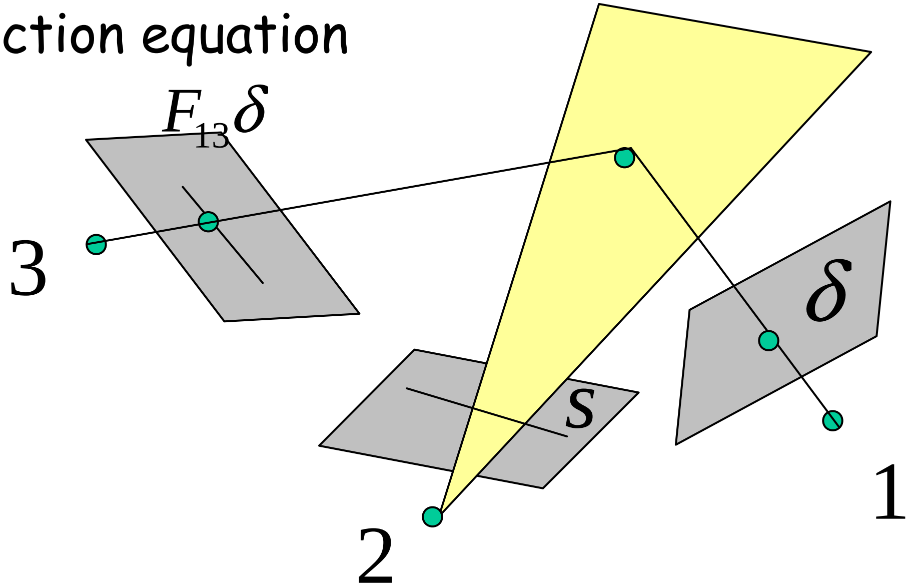
Slices of the Trifocal Tensor

$\delta_j T_i^{jk}$ is the homography matrix from 1 to 3 induced by the plane defined by the image line δ and the second camera center.

$$\delta^i T_i^{jk} = ?$$

$\delta^i s_j T_i^{jk}$ is the reprojection equation

The result is a point on the epipolar line of δ on image 3



Slices of the Trifocal Tensor

$$\delta^i T_i^{\ jk} = G^{jk}$$

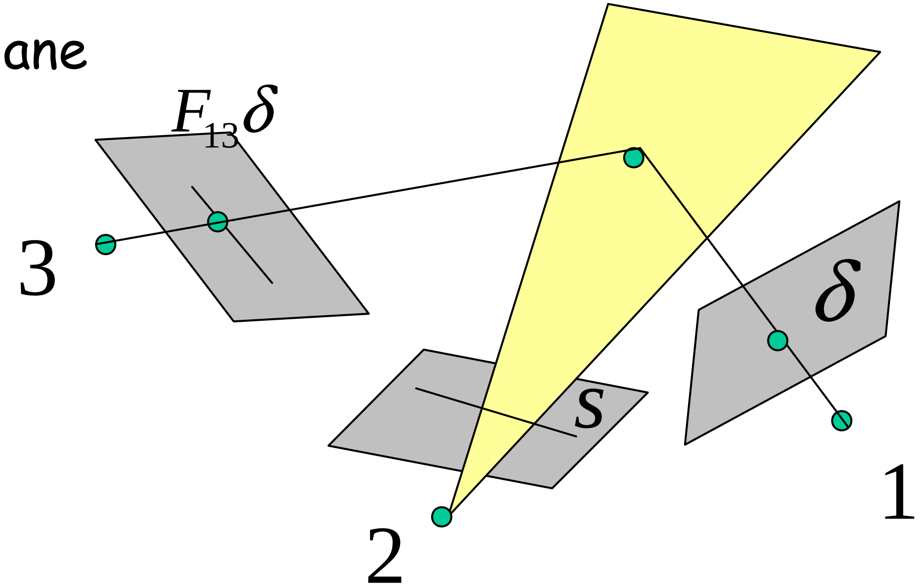
Gs Is a point on the epipolar line $F_{13}\delta$

→ $rank(G) = 2$

(because it maps the dual plane onto collinear points)

$$null(G) = F_{12}\delta$$

$$null(G^T) = F_{13}\delta$$



18 Parameters for the Trifocal Tensor

$$T_i^{\ jk} = e'^j \, b_i^k - e''^k \, a_i^j$$

$$e'^j \, (b_i^k + n_i e''^k) - e''^k \, (a_i^j + n_i e'^j) \qquad \forall n$$

$$= T_i^{\ jk} + n_i e'^j \, e''^k - n_i e'^j \, e''^k$$

$$= T_i^{\ jk}$$

$T_i^{\ jk}$ Has 24 parameters (9+9+3+3)
minus 1 for global scale
minus 2 for scaling e’,e” to be unit vectors
minus 3 for setting n_i such that B has a vanishing column

= 18 independent parameters

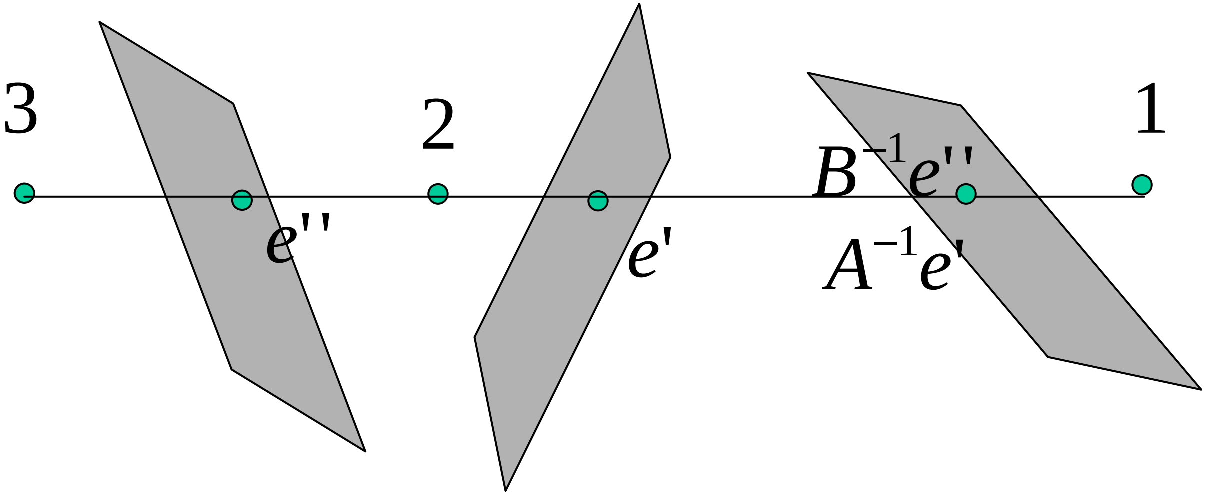
➡ We should expect to find 9 non-linear constraints among the 27 entries of the tensor (admissibility constraints).

18 Parameters for the Trifocal Tensor

What happens when the 3 camera centers are collinear?

(we saw that pairwise F-mats account for 15 parameters).

$$A^{-1}e' \cong B^{-1}e''$$



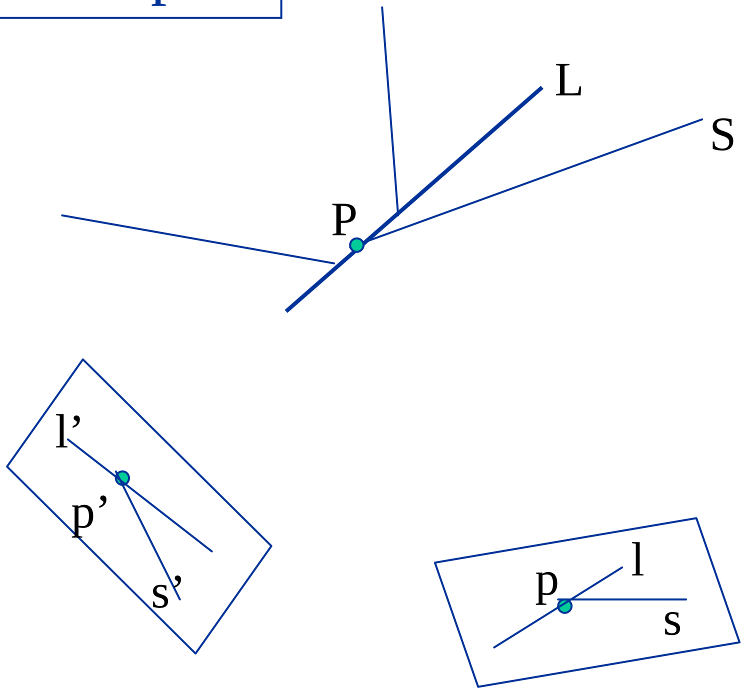
This provides two additional (non-linear) constraints, thus 18-2=16.

Items not Covered

- Degenerate configurations (Linear Line Complex, Quartic Curve)
- The source of the 9 admissibility constraints (come from the homography slices).
- Concatenation of trifocal tensors along a sequence

Quadrifocal Tensor

Linear Line Complex



$$H_\pi p \cong p'$$

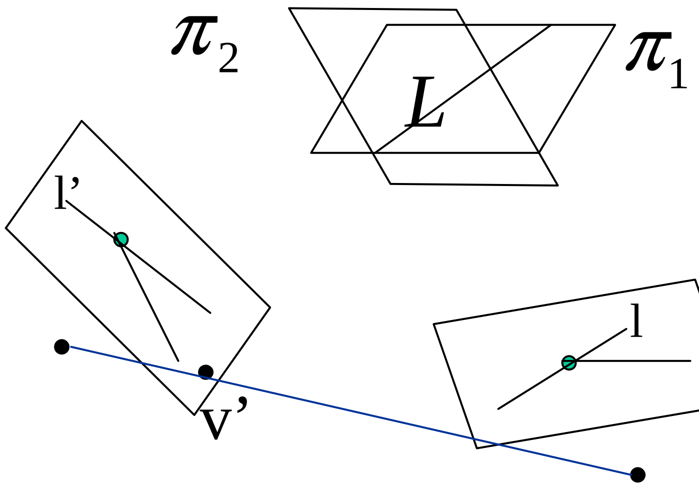
Where π is any plane coincident with L

$$p \cong (l \quad \times \quad s) = [l]_x s$$

$$\Rightarrow \qquad s'^T H_\pi [l]_x s = s'^T G_\pi s = 0$$

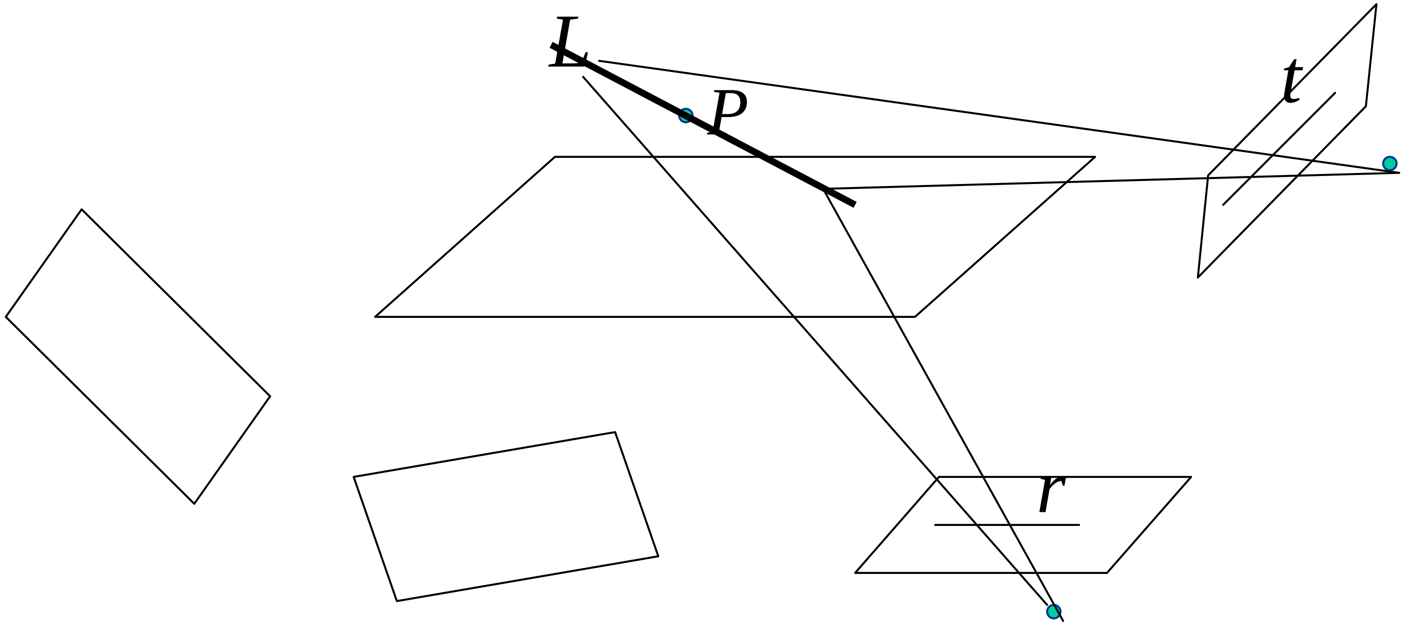
For **all** lines s passing through p and **all** lines s’ passing through p’

G_π is unique



$$H_{\pi 2} = \lambda H_{\pi 1} + v' l^T$$

$$G_{\pi 2} = (\lambda H_{\pi 1} + v' l^T)[l]_x \cong H_{\pi 1}[l]_x = G_{\pi 1}$$

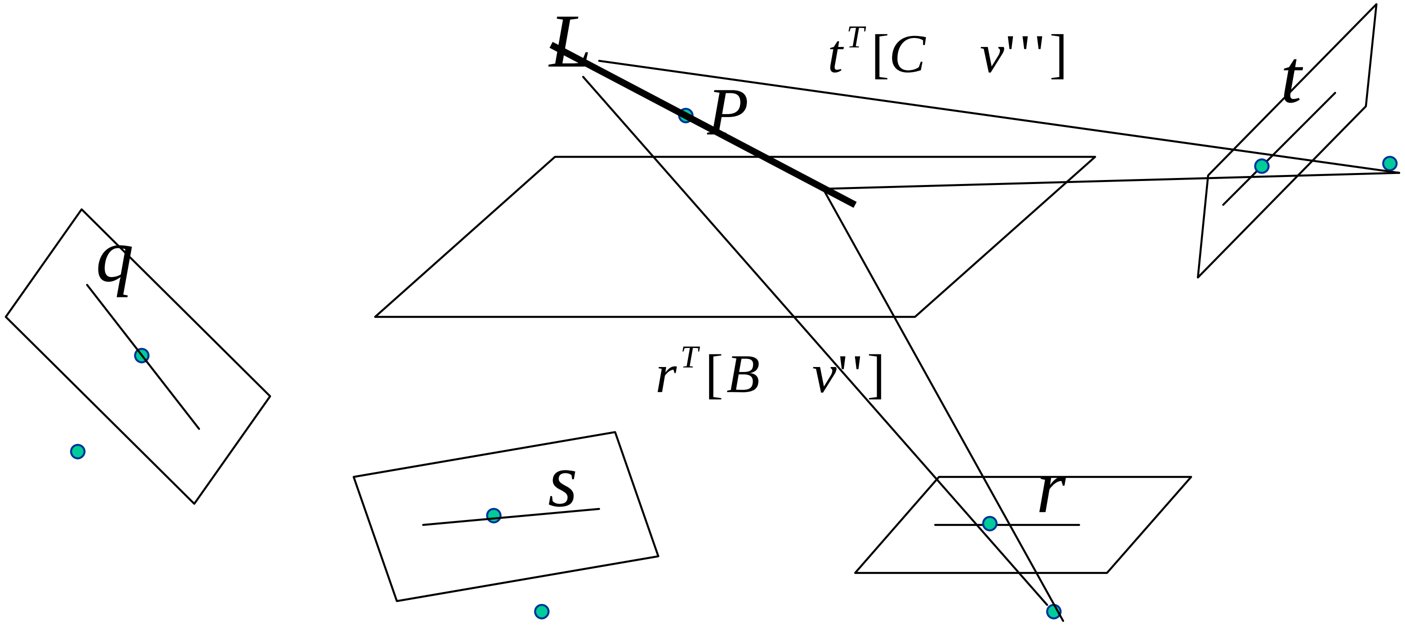


$$p \cong [I \quad 0]P$$

$$p' \cong [A \quad v']P$$

$$p'' \cong [B \quad v'']P$$

$$p''' \cong [C \quad v''']P$$

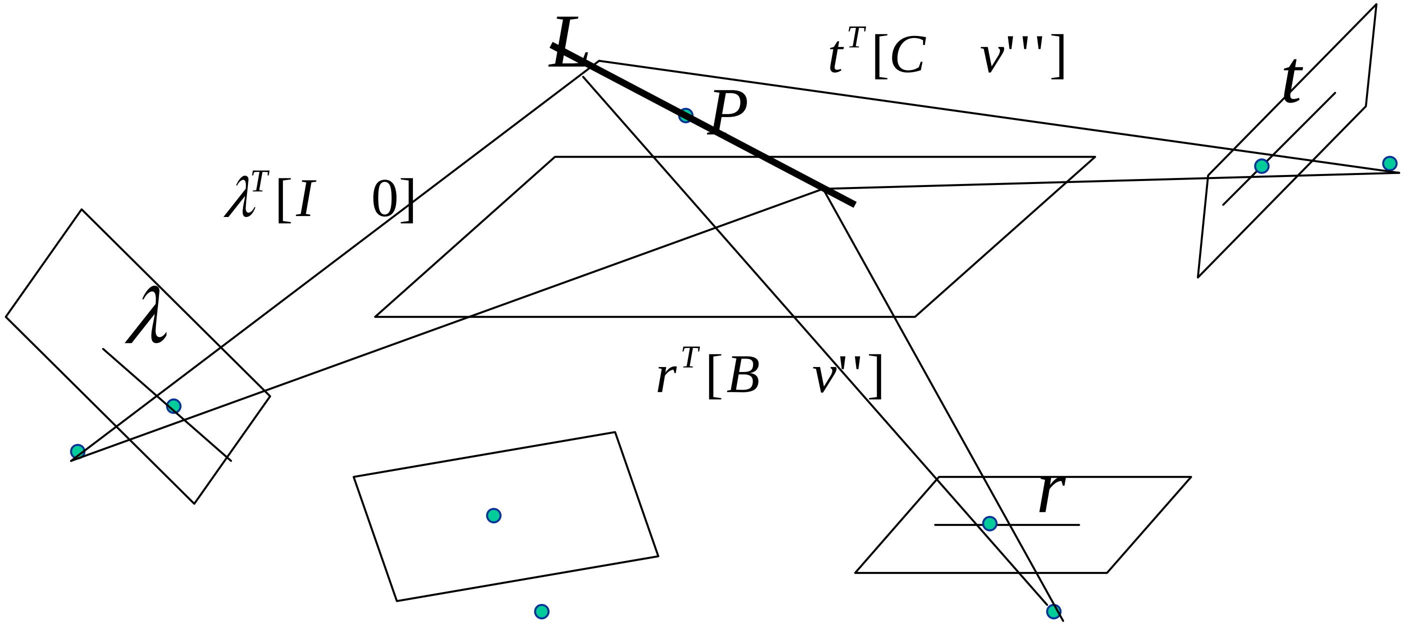


We wish to construct an LLC mapping Q(r,t) whose kernel is L

$$s^T Q(r,t) q = 0$$

$$\Longrightarrow \quad q_i s_j r_k t_l Q^{ijkl} = 0$$

For all lines s passing through p’ and all lines q passing through p



$$\begin{pmatrix} \lambda \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} B^T r \\ v''^T r \end{pmatrix} + \beta \begin{pmatrix} C^T t \\ v'''^T t \end{pmatrix}$$

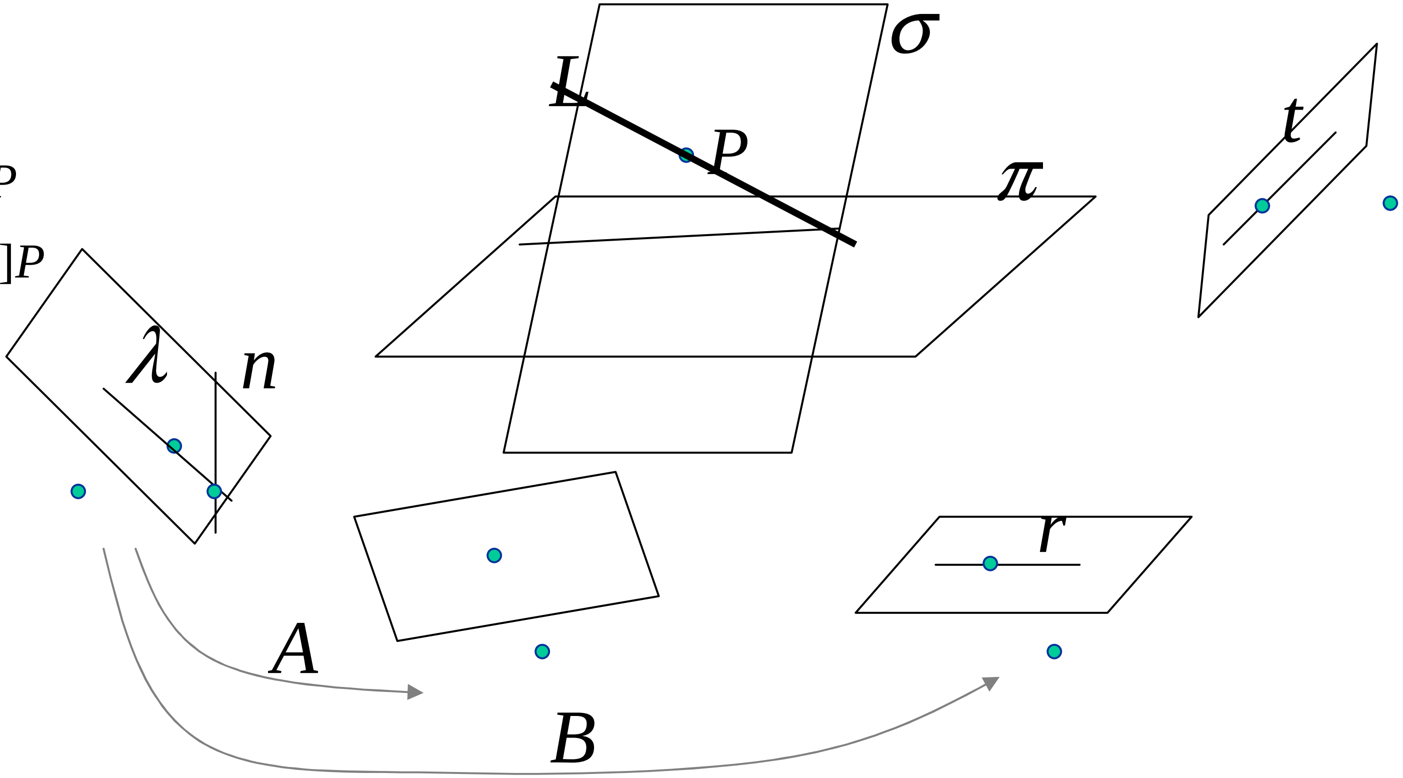
⇨ $$\lambda \cong (v'''^T t) B^T r - (v''^T r) C^T t$$

$$p \cong [I \quad 0]P$$

$$p' \cong [A \quad v']P$$

$$p'' \cong [B \quad v'']P$$

$$p''' \cong [C \quad v''']P$$



$$A_\sigma \cong A + v' n^T$$

$$Q(r,t) = A_\sigma [\lambda]_\times \cong A[\lambda]_\times + v'(n \times \lambda)^T$$

$$(n \times \lambda) \cong B^T r \quad \times \quad C^T t$$

$$(n\times\lambda)\cong B^T r \quad \times \quad C^T t$$

$$\lambda\cong(v'''^T\, t)B^T r-(v''^T\, r)C^T t$$

$$Q(r,t)=A_\sigma[\lambda]_\times\cong A[\lambda]_\times+v'(n\times\lambda)^T$$

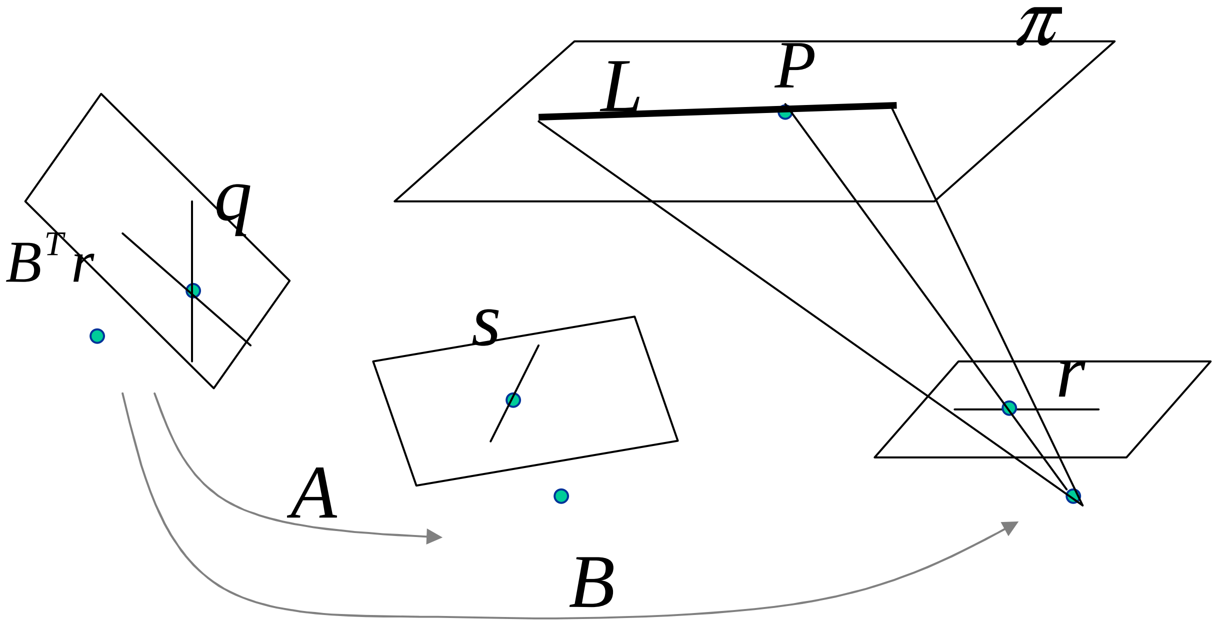
$$\Downarrow$$

$$Q(r,t)=v'(B^T r\times C^T t)-(v''^T\, r)A[C^T t]_\times+(v'''^T\, t)A[B^T r]_\times$$

$$s^T Q(r,t)q=0$$

$$\Longrightarrow \quad (s^T v')(B^T r\times C^T t)q-(v''^T\, r)s^T A[C^T t]_\times q+(v'''^T\, t)s^T A[B^T r]_\times q=0$$

$$(s^T v')(B^T r \times C^T t)q - (v''^T r)s^T A[C^T t]_\times q + (v'''^T t)\boxed{s^T A[B^T r]_\times q} = 0$$



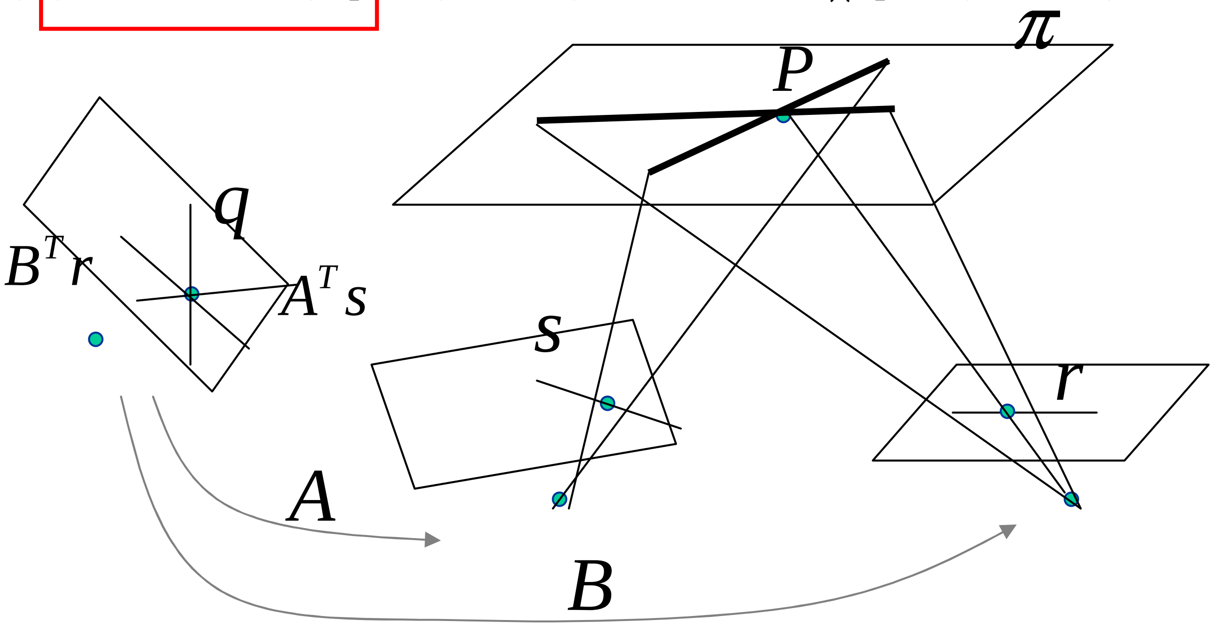
LLC between views 1,2 with kernel L is:

$$A[B^T r]_\times$$

$$\longrightarrow \qquad s^T A[B^T r]_\times q = 0$$

For all q,s,r through p,p’,p’’

$$(s^T v')\boxed{(B^T r \times C^T t)q} - (v''^T r) s^T A[C^T t]_\times q + (v'''^T t) s^T A[B^T r]_\times q = 0$$



$$rank\left[q \quad A^T s \quad B^T r\right] = 2$$

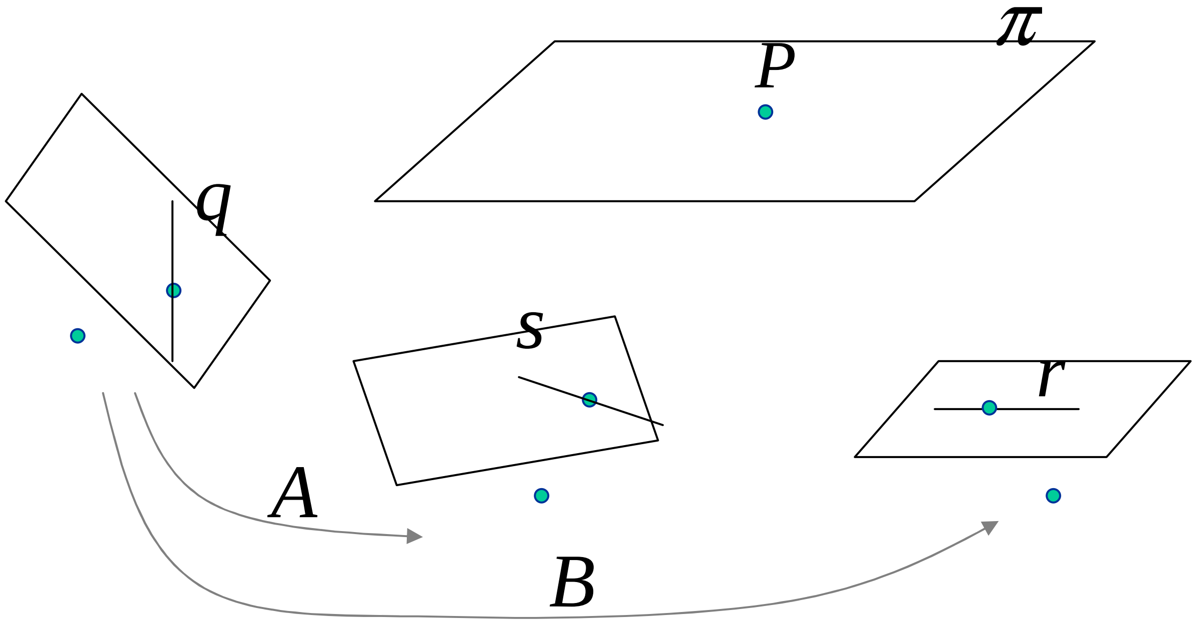
For all q,s,r through p,p’,p’’

$$\Downarrow$$

$$q^T (B^T r \quad \times \quad A^T s) = 0$$

$$s^T A[B^T r]_\times q = 0$$

Dual Homography Tensor



$$q^T (B^T r \quad \times \quad A^T s) = 0$$

$$\Longrightarrow \qquad q_i s_j r_k (\varepsilon^{iun} a_u^j b_n^k) = 0$$

$$q_i s_j r_k H^{ijk} = 0$$

Quadrifocal Tensor

$$(s^T v')(B^T r \times C^T t)q - (v''^T r)s^T A[C^T t]_\times q + (v'''^T t)s^T A[B^T r]_\times q = 0$$

$$\Downarrow$$

$$(s_j v'^j)H^{ikl} q_i r_k t_l - (r_k v''^k)H^{ijl} q_i s_j t_l + (t_l v'''^l)H^{ijk} q_i s_j r_k = 0$$

$$\Downarrow$$

$$q_i s_j r_k t_l Q^{ijkl} = 0$$

$$\boxed{Q^{ijkl} = v'^j H^{ikl} - v''^k H^{ijl} + v'''^l H^{ijk}}$$

$T_i^{jk} = v'^j b_i^k - v''^k a_i^j$ The trilinear (trifocal) tensor

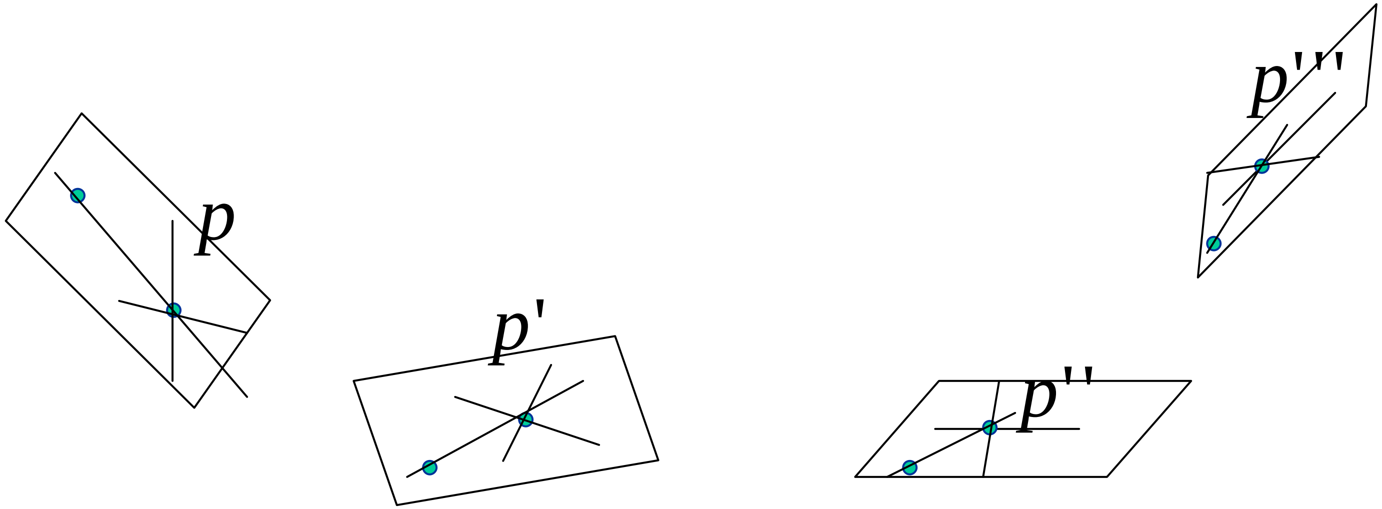
- Gauge Invariance

$$[I \quad 0], [A + v' w^T \quad v'], [B + v'' w^T \quad v''], [C + v''' w^T \quad v''']$$

$$q_i s_j r_k t_l \overline{Q}^{ijkl} = 0$$

$$Q^{ijkl} = \overline{Q}^{ijkl}$$ For all choices of w

How Many matching points are needed?

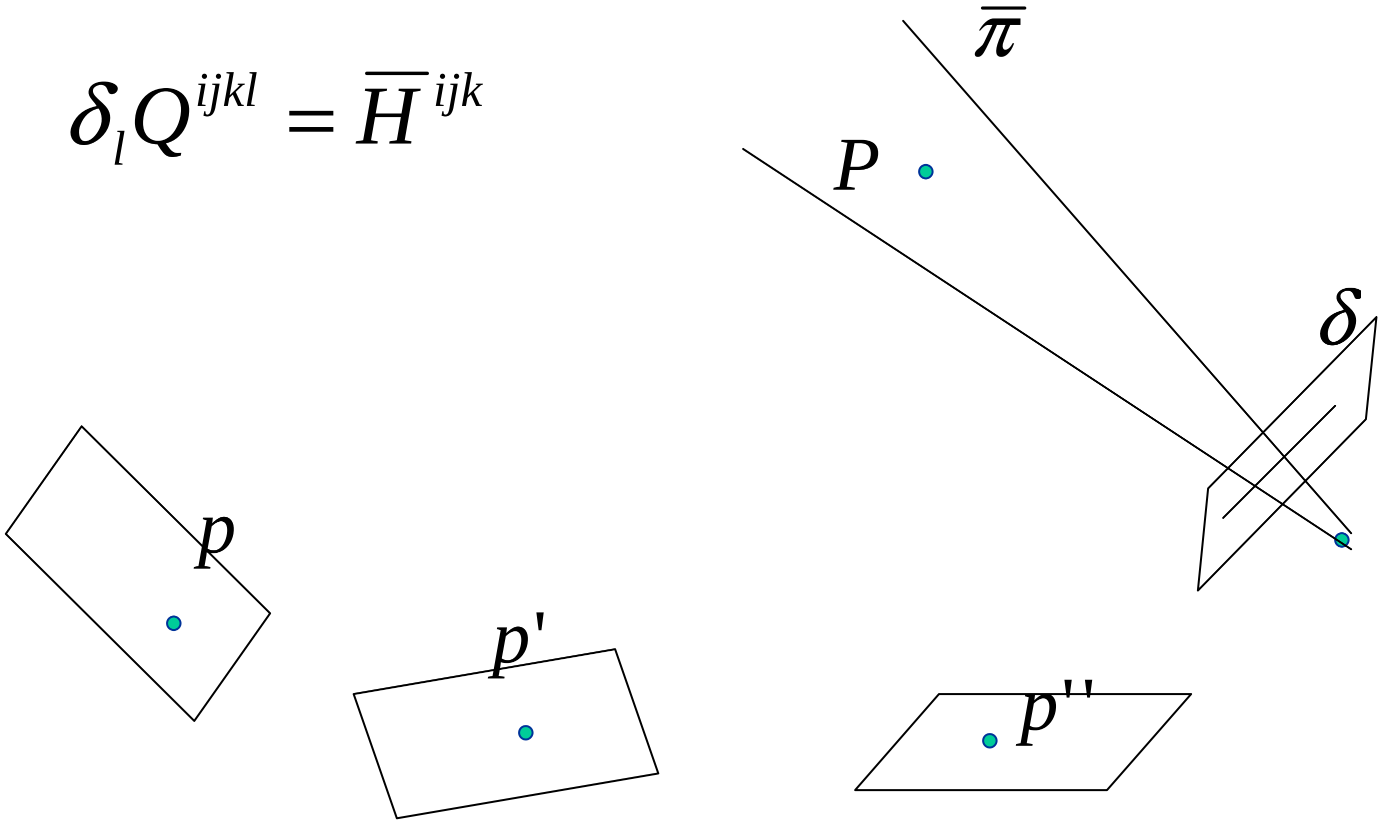


$$q_i^{\eta} s_j^{\mu} r_k^{\nu} t_l^{\sigma} Q^{ijkl} = 0 \qquad \eta, \mu, \nu, \sigma = 1,2$$

- 1st point: 16 quadlinear equations for Q
- 2nd point: 15 equations
- 3rd point: 14 equations
- ……

Single Contraction

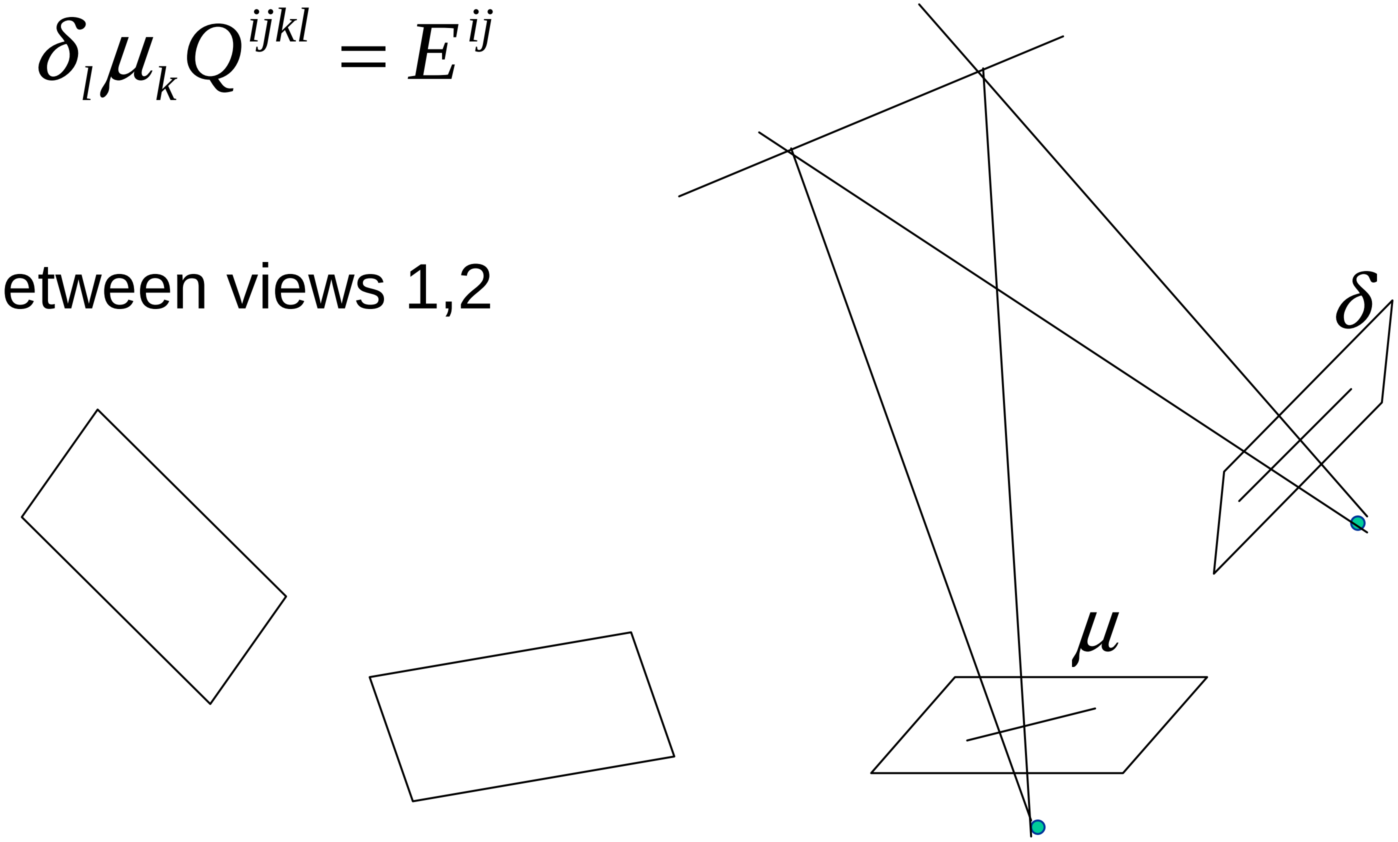
$$\delta_l Q^{ijkl} = \overline{H}^{ijk}$$



Double Contraction

$$\delta_l \mu_k Q^{ijkl} = E^{ij}$$

LLC between views 1,2



Triple Contraction

$$s_j r_k t_l Q^{ijkl} \cong p^i$$

Items not Covered

- Quad constructed from Trifocal and Fundamental Matrix
- Fundamental Matrix from Quadrifocal
- Trifocal from Quadrifocal
- Projection Matrices from Quadrifocal
- 51 Non-linear Constraints