On the tradeoff between computational complexity and sample complexity in learning

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PAC Learning

- $\mathcal{X}$ - domain set

Learning (informally): Use $S$ to find some $h \approx h^{\star}$

Learning (formally):

$D$ - a distribution over $\mathcal{X}$

Assumption: instances of $S$ are chosen i.i.d. from $D$

Error: $\text{err}(h) = P[h(x) \neq h^{\star}(x)]$

Goal: use $S$ to find $h$ s.t. w.p. $1 - \delta$, $\text{err}(h) \leq \epsilon$
PAC Learning

- \( \mathcal{X} \) - domain set
- \( \mathcal{Y} = \{\pm 1\} \) - target set

Predictor: \( h: \mathcal{X} \rightarrow \mathcal{Y} \)

Training set: \( S = (x_1, h^\star(x_1)), \ldots, (x_m, h^\star(x_m)) \)

Learning (informally): Use \( S \) to find some \( h \approx h^\star \)

Learning (formally)

- \( D \) - a distribution over \( \mathcal{X} \)
- Assumption: instances of \( S \) are chosen i.i.d. from \( D \)

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Goal: use \( S \) to find \( h \) s.t. w.p. 1 - \( \delta \), \( \text{err}(h) \leq \epsilon \)

Prior knowledge: \( h^\star \in H \)
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- Prior knowledge: $h^* \in \mathcal{H}$
Sample complexity — How many examples are needed?
Complexity of Learning

- **Sample complexity** — How many examples are needed?
  - Vapnik: exactly $\frac{\text{VC}(\mathcal{H}) \log(1/\delta)}{\epsilon}$

- Computational complexity — How much time is needed?
  - Naively: it takes $\Omega(|\mathcal{H}|)$ to implement the ERM

- Exponential gap between time and sample complexity (?)

- This talk — joint time-sample dependency

$$
\text{err}(m', \tau) \overset{\text{def}}{=} \min_{m \leq m'} \min_{A} \text{time}(A) \leq \tau \mathbb{E}[\text{err}(A(S))]
$$

- Sample complexity — $\arg\min \{m': \text{err}(m', \infty) \leq \epsilon\}$

- Data laden — $\text{err}(\infty, \tau)$
Sample complexity — How many examples are needed?

- Vapnik: exactly $\frac{\text{VC}(\mathcal{H}) \log(1/\delta)}{\epsilon}$
- Using the ERM (empirical risk minimization)
Complexity of Learning

- **Sample complexity** — How many examples are needed?
  - Vapnik: exactly $\frac{V\!C(H) \log(1/\delta)}{\epsilon}$
  - Using the ERM (empirical risk minimization)

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  \]

- Sample complexity — $\arg \min \{m' : \text{err}(m', \infty) \leq \epsilon\}$
- Data laden — err($\infty$, $\tau$)
Main Question

How much time, $\tau$, is needed to achieve error $\leq \epsilon$ as a function of sample size, $m$?
Warmup example

\[ \mathcal{X} = \{0, 1\}^d \]
Warmup example

- $\mathcal{X} = \{0, 1\}^d$
- $\mathcal{H}$ is 3-term DNF formulae:

$$h(x) = T_1(x) \lor T_2(x) \lor T_3(x),$$
where each $T_i$ is a conjunction.

- $|\mathcal{H}| \leq 3^d$
- Therefore sample complexity is order $d/\epsilon$.

Kearns & Vazirani: If $\text{RP} \neq \text{NP}$, it is not possible to efficiently find $h \in \mathcal{H}$ s.t. $\text{err}(h) \leq \epsilon$.

Claim: if $m \geq d^3/\epsilon$ it is possible to find a predictor with error $\leq \epsilon$ in polynomial time.
Warmup example

- $\mathcal{X} = \{0, 1\}^d$
- $\mathcal{H}$ is 3-term DNF formulae:
  - $h(x) = T_1(x) \vee T_2(x) \vee T_3(x)$, where each $T_i$ is a conjunction

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\[ |H| \leq 3 \]

Therefore, sample complexity is order $d/\epsilon$.

Kearns & Vazirani: If RP $\neq$ NP, it is not possible to efficiently find $h \in \mathcal{H}$ s.t. $\text{err}(h) \leq \epsilon$

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Warmup example

- \( \mathcal{X} = \{0, 1\}^d \)
- \( \mathcal{H} \) is 3-term DNF formulae:
  - \( h(x) = T_1(x) \lor T_2(x) \lor T_3(x) \), where each \( T_i \) is a conjunction
  - E.g. \( h(x) = (x_1 \land \neg x_3 \land x_7) \lor (x_4 \land x_2) \lor (x_5 \land \neg x_9) \)

\[ |H| \leq 3 \]

Therefore sample complexity is order \( \frac{d}{\epsilon} \)

Kearns & Vazirani: If \( \text{RP} \neq \text{NP} \), it is not possible to efficiently find \( h \in \mathcal{H} \) s.t. err(\( h \)) \( \leq \epsilon \)

Claim: if \( m \geq \frac{d}{3} \frac{3}{\epsilon} \) it is possible to find a predictor with error \( \leq \epsilon \) in polynomial time
Warmup example

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  - $|\mathcal{H}| \leq 3^{3d}$ therefore sample complexity is order $d/\epsilon$
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$\tau$ $\mathcal{M}$
Warmup example

- $\mathcal{X} = \{0, 1\}^d$
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- Kearns & Vazirani: If $\mathsf{RP} \neq \mathsf{NP}$, it is not possible to efficiently find $h \in \mathcal{H}$ s.t. $\text{err}(h) \leq \epsilon$
- **Claim:** if $m \geq d^3/\epsilon$ it is possible to find a predictor with error $\leq \epsilon$ in polynomial time
Observation: $T_1 \lor T_2 \lor T_3 = \land_{u \in T_1, v \in T_2, w \in T_3} (u \lor v \lor w)$

Define: $\psi : \mathcal{X} \to \{0, 1\}^{2(2d)^3}$ s.t. for each triplet of literals $u, v, w$ there are two variables indicating if $u \lor v \lor w$ is true or false

Observation: Exists Halfspace s.t. $h^*(x) = \text{sgn} \left( \langle w, \psi(x) \rangle + b \right)$

Therefore, can solve ERM w.r.t. Halfspaces (linear programming)

VC dimension of Halfspaces is the dimension

Sample complexity is order $d^3 / \epsilon$
Trading samples for runtime

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<th>samples</th>
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<tr>
<td>3-DNF</td>
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<td>$2^d$</td>
</tr>
<tr>
<td>Halfspace</td>
<td>$\frac{d^3}{\epsilon}$</td>
<td>poly$(d)$</td>
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The diagram illustrates the trade-off between time ($\tau$) and data ($m$) for 3-DNF and Halfspace algorithms.
But,

- The lower bound on the computational complexity is only for *proper* learning — there’s no lower bound on the computational complexity of improper learning with $d/\epsilon$ examples.
- The lower bound on the sample complexity of Halfspaces is in the general case — here we have a specific structure.

The interesting questions:

- Is the curve really true? Can one construct 'correct' lower bounds?
- If the curve is true, one should be able to construct more algorithms on the curve. How?
Second example: Online Ads Placement

For $t = 1, 2, \ldots, m$

- Learner receives side information $x_t \in \mathbb{R}^d$
- Learner predicts $\hat{y}_t \in [k]$
- Learner pay cost $1[\hat{y}_t \neq h^*(x_t)]$
- “Bandit setting” — learner does not know $h^*(x_t)$

**Goal:** Minimize error rate:

$$
\text{err} = \frac{1}{m} \sum_{t=1}^{m} 1[\hat{y}_t \neq h^*(x_t)].
$$
\[ \mathcal{H} = \{ x \mapsto \arg\max_r (W x)_r : W \in \mathbb{R}^{k,d}, \|W\|_F \leq 1 \} \]
Assumption: Data is separable with margin $\mu$:

$$\forall t, \forall r \neq y_t, (Wx_t)_y - (Wx_t)_r \geq \mu$$
First approach – Halving

Halving for Bandit Multiclass categorization

Initialize: \( V_1 = \mathcal{H} \)

For \( t = 1, 2, \ldots \)

- Receive \( x_t \)
- For all \( r \in [k] \) let \( V_t(r) = \{ h \in V_t : h(x_t) = r \} \)
- Predict \( \hat{y}_t \in \text{arg max}_r |V_t(r)| \)
- If \( 1[\hat{y}_t \neq y_t] \) set \( V_{t+1} = V_t \setminus V_t(\hat{y}_t) \)

Analysis:
Whenever we err \( |V_{t+1}| \leq (1 - \frac{1}{k}) |V_t| \leq \exp(-\frac{1}{k}) |V_t| \)
Therefore: err \( \leq k \log(|\mathcal{H}| / m) \)
Equivalently, sample complexity is \( k \log(|\mathcal{H}| / \epsilon) \)
First approach – Halving

Halving for Bandit Multiclass categorization

Initialize: $V_1 = \mathcal{H}$
For $t = 1, 2, \ldots$
  
  - Receive $x_t$
  
  - For all $r \in [k]$ let $V_t(r) = \{ h \in V_t : h(x_t) = r \}$
  
  - Predict $\hat{y}_t \in \arg \max_r |V_t(r)|$
  
  - If $1[\hat{y}_t \neq y_t]$ set $V_{t+1} = V_t \setminus V_t(\hat{y}_t)$

Analysis:

- Whenever we err $|V_{t+1}| \leq (1 - \frac{1}{k^2}) |V_t| \leq \exp(-1/k) |V_t|
- Therefore: err $\leq \frac{k \log(|\mathcal{H}|)}{m}$
- Equivalently, sample complexity is $\frac{k \log(|\mathcal{H}|)}{\epsilon}$
Using Halving

- Step 1: Dimensionality reduction to \( d' = O\left(\frac{\ln(m+k)}{\mu^2}\right) \)
- Step 2: Discretize \( \mathcal{H} \) to \( (1/\mu)^{kd'} \) hypotheses
- Apply Halving on the resulting finite set of hypotheses
Using Halving

- Step 1: Dimensionality reduction to $d' = O\left(\frac{\ln(m+k)}{\mu^2}\right)$
- Step 2: Discretize $\mathcal{H}$ to $(1/\mu)^{kd'}$ hypotheses
  - Apply Halving on the resulting finite set of hypotheses

Analysis:

- Sample complexity is order of $\frac{k^2/\mu^2}{\epsilon}$
- But runtime grows like $(1/\mu)^{kd'} = (m + k)^{\tilde{O}(k/\mu^2)}$
How can we improve runtime?

- Halving is not efficient because it does not utilize the structure of $\mathcal{H}$
- In the full information case: Halving can be made efficient because each version space $V_t$ can be made convex!
- The Perceptron is a related approach which utilizes convexity and works in the full information case
- Next approach: Let's try to rely on the Perceptron
The Multiclass Perceptron

For $t = 1, 2, \ldots, m$

- Receive $x_t \in \mathbb{R}^d$
- Predict $\hat{y}_t = \arg\max_r (W^t x_t)_r$
- Receive $y_t = h^*(x_t)$
- If $\hat{y}_t \neq y_t$ update: $W^{t+1} = W^t + U^t$
The Multiclass Perceptron

For \( t = 1, 2, \ldots, m \)
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- Receive \( y_t = h^*(\mathbf{x}_t) \)
- If \( \hat{y}_t \neq y_t \) update: \( W^{t+1} = W^t + U^t \)

\[
U^t = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\cdots & \mathbf{x}_t & \cdots \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\cdots & -\mathbf{x}_t & \cdots \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}
\]

Row \( \hat{y}_t \)
Row \( y_t \)

Problem: In the bandit case, we’re blind to value of \( y_t \)
Explore: From time to time, instead of predicting $\hat{y}_t$ guess some $\tilde{y}_t$

Suppose we get the feedback ’correct’, i.e. $\tilde{y}_t = y_t$

Then, we have full information for Perceptron’s update:

$$(x_t, \hat{y}_t, \tilde{y}_t = y_t)$$
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$$(x_t, \hat{y}_t, \tilde{y}_t = y_t)$$

Exploration-Exploitation Tradeoff:
- When exploring we may have $\tilde{y}_t = y_t \neq \hat{y}_t$ and can learn from this
- When exploring we may have $\tilde{y}_t \neq y_t = \hat{y}_t$ and then we had the right answer in our hands but didn't exploit it
The Banditron (Kakade, S, Tewari 08)

For $t = 1, 2, \ldots, m$

- Receive $\mathbf{x}_t \in \mathbb{R}^d$
- Set $\hat{y}_t = \arg \max_r (W^t \mathbf{x}_t)_r$
- Define: $P(r) = (1 - \gamma) \mathbf{1}[r = \hat{y}_t] + \frac{\gamma}{k}$
- Randomly sample $\tilde{y}_t$ according to $P$
- Predict $\tilde{y}_t$
- Receive feedback $\mathbf{1}[\tilde{y}_t = y_t]$
- Update: $W^{t+1} = W^t + \tilde{U}^t$
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For \( t = 1, 2, \ldots, m \)
- Receive \( x_t \in \mathbb{R}^d \)
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- Randomly sample \( \tilde{y}_t \) according to \( P \)
- Predict \( \tilde{y}_t \)
- Receive feedback \( \mathbf{1}[\tilde{y}_t = y_t] \)
- Update: \( W^{t+1} = W^t + \tilde{U}^t \) where

\[
\tilde{U}^t = \begin{bmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\end{bmatrix}
\]

Row \( \hat{y}_t \)
Row \( \tilde{y}_t \)
Theorem

- Banditron’s sample complexity is order of \( \frac{k/\mu^2}{\epsilon^2} \)
- Banditron’s runtime is \( O(k/\mu^2) \)
The Banditron (Kakade, S, Tewari 08)

Theorem

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The crux of difference between Halving and Banditron:

- Without having the full information, the version space is non-convex and therefore it is hard to utilize the structure of \( \mathcal{H} \)
- Because we relied on the Perceptron we did utilize the structure of \( \mathcal{H} \) and got an efficient algorithm
- We managed to obtain 'full-information examples' by using exploration
- The price of exploration is a higher regret
### Trading samples for runtime

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<td>$k/\mu^2$</td>
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#### Diagram

- **$\tau$**
- **Halving**
- **Banditron**

$m$
Agnostic PAC:

- $\mathcal{D}$ - arbitrary distribution over $\mathcal{X} \times \mathcal{Y}$
- Training set: $S = (x_1, y_1), \ldots, (x_m, y_m)$
- Goal: use $S$ to find $h_S$ s.t. w.p. $1 - \delta$,

$$\text{err}(h_S) \leq \min_{h \in \mathcal{H}} \text{err}(h) + \epsilon$$
Hypothesis class

\[ \mathcal{H} = \{ \mathbf{x} \mapsto \phi(\langle \mathbf{w}, \mathbf{x} \rangle) : \| \mathbf{w} \|_2 \leq 1 \}, \quad \phi(z) = \frac{1}{1 + \exp(-z/\mu)} \]

- Probabilistic classifier: \( \mathbb{P}[h_w(\mathbf{x}) = 1] = \phi(\langle \mathbf{w}, \mathbf{x} \rangle) \)
- Loss function: \( \text{err}(\mathbf{w}; (\mathbf{x}, y)) = \mathbb{P}[h_w(\mathbf{x}) \neq y] = \left| \phi(\langle \mathbf{w}, \mathbf{x} \rangle) - \frac{y + 1}{2} \right| \)
- Remark: Dimension can be infinite (kernel methods)
First approach — sub-sample covering

- **Claim:** exists $1/(\epsilon \mu^2)$ examples from which we can efficiently learn $w^*$ up to error of $\epsilon$
- **Proof idea:**
  - $S' = \{(x_i, y'_i) : y'_i = y_i$ if $y_i \langle w^*, x_i \rangle < -\mu$ and else $y'_i = -y_i\}$
  - Use surrogate convex loss $\frac{1}{2} \max\{0, 1 - y \langle w, x \rangle / \gamma\}$
  - Minimizing surrogate loss on $S' \Rightarrow$ minimizing original loss on $S$
  - Sample complexity w.r.t. surrogate loss is $1/(\epsilon \mu^2)$

**Analysis**

- **Sample complexity:** $1/(\epsilon \mu)^2$
- **Time complexity:** $m^{1/(\epsilon \mu^2)} = \left(\frac{1}{\epsilon \mu}\right)^{1/(\epsilon \mu^2)}$
Learning fuzzy halfspaces using Infinite-Dimensional-Polynomial-Kernel

- **Original class:** \( \mathcal{H} = \{ x \mapsto \phi(\langle w, x \rangle) \} \)
Learning fuzzy halfspaces using Infinite-Dimensional-Polynomial-Kernel

- **Original class**: \( \mathcal{H} = \{ \mathbf{x} \mapsto \phi(\langle \mathbf{w}, \mathbf{x} \rangle) \} \)
- **Problem**: Loss is non-convex w.r.t. \( \mathbf{w} \)
Second Approach – IDPK (S, Shamir, Sridharan)

Learning fuzzy halfspaces using **Infinite-Dimensional-Polynomial-Kernel**

- **Original class:** $\mathcal{H} = \{ \mathbf{x} \mapsto \phi(\langle \mathbf{w}, \mathbf{x} \rangle) \}$
- **Problem:** Loss is non-convex w.r.t. $\mathbf{w}$
- **Main idea:** Work with a larger hypothesis class for which the loss becomes convex
Step 2 – Learning fuzzy halfspaces with IDPK

- Original class: \( \mathcal{H} = \{ x \mapsto \phi(\langle w, x \rangle) : \|w\| \leq 1 \} \)
- New class: \( \mathcal{H}' = \{ x \mapsto \langle v, \psi(x) \rangle : \|v\| \leq B \} \) where \( \psi : \mathcal{X} \rightarrow \mathbb{R}^N \) s.t.
  \[ \forall j, \forall (i_1, \ldots, i_j), \psi(x)(i_1, \ldots, i_j) = 2^{j/2} x_{i_1} \cdots x_{i_j} \]
Step 2 – Learning fuzzy halfspaces with IDPK

- Original class: \( \mathcal{H} = \{ \mathbf{x} \mapsto \phi(\langle \mathbf{w}, \mathbf{x} \rangle) : \| \mathbf{w} \| \leq 1 \} \)

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  \[
  \forall j, \forall (i_1, \ldots, i_j), \psi(\mathbf{x})(i_1, \ldots, i_j) = 2^{j/2} x_{i_1} \cdots x_{i_j}
  \]

Lemma (S, Shamir, Sridharan 2009)

If \( B = \exp(\tilde{O}(1/\mu)) \) then for all \( h \in \mathcal{H} \) exists \( h' \in \mathcal{H}' \) s.t. for all \( \mathbf{x} \),
\( h(\mathbf{x}) \approx h'(\mathbf{x}) \).
Step 2 – Learning fuzzy halfspaces with IDPK

- **Original class:** $\mathcal{H} = \{ x \mapsto \phi(\langle w, x \rangle) : \|w\| \leq 1 \}$
- **New class:** $\mathcal{H}' = \{ x \mapsto \langle v, \psi(x) \rangle : \|v\| \leq B \}$ where $\psi : \mathcal{X} \rightarrow \mathbb{R}^N$ s.t. $\forall j, \forall (i_1, \ldots, i_j), \psi(x)_{(i_1,\ldots,i_j)} = 2^{j/2} x_{i_1} \cdots x_{i_j}$

**Lemma (S, Shamir, Sridharan 2009)**

*If $B = \exp(\tilde{O}(1/\mu))$ then for all $h \in \mathcal{H}$ exists $h' \in \mathcal{H}'$ s.t. for all $x$, $h(x) \approx h'(x)$.***

**Remark:** The above is a pessimistic choice of $B$. In practice, smaller $B$ suffices. Is it tight? Even if it is, are there natural assumptions under which a better bound holds?

(e.g. Kalai, Klivans, Mansour, Servedio 2005)
Proof idea

- Polynomial approximation: \( \phi(z) \approx \sum_{j=0}^{\infty} \beta_j z^j \)
Proof idea

- Polynomial approximation: $\phi(z) \approx \sum_{j=0}^{\infty} \beta_j z^j$
- Therefore:

$$\phi(\langle w, x \rangle) \approx \sum_{j=0}^{\infty} \beta_j (\langle w, x \rangle)^j$$

$$= \sum_{j=0}^{\infty} \sum_{k_1, \ldots, k_j} 2^{-j/2} \beta_j 2^{j/2} w_{k_1} \cdots w_{k_j} x_{k_1} \cdots x_{k_j}$$

$$= \langle v_w, \psi(x) \rangle$$
Proof idea

- Polynomial approximation: \( \phi(z) \approx \sum_{j=0}^{\infty} \beta_j z^j \)

- Therefore:

\[
\phi(\langle w, x \rangle) \approx \sum_{j=0}^{\infty} \beta_j (\langle w, x \rangle)^j
= \sum_{j=0}^{\infty} \sum_{k_1, \ldots, k_j} 2^{-j/2} \beta_j 2^{j/2} w_{k_1} \cdots w_{k_j} x_{k_1} \cdots x_{k_j}
= \langle v_w, \psi(x) \rangle
\]

- To obtain a concrete bound we use **Chebyshev approximation technique**: Family of orthogonal polynomials w.r.t. inner product:

\[
\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} \, dx
\]
Although the dimension is infinite, can be solved using the kernel trick.

The corresponding kernel (a.k.a. Vovk's infinite polynomial):

$$\langle \psi(x), \psi(x') \rangle = K(x, x') = \frac{1}{1 - \frac{1}{2}\langle x, x' \rangle}$$

Algorithm boils down to linear regression with the above kernel.

Convex! Can be solved efficiently.

Sample complexity: \((B/\epsilon)^2 = 2\tilde{O}(1/\mu)/\epsilon^2\)

Time complexity: \(m^2\)
### Trading samples for time

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>sample</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covering</td>
<td>$\frac{1}{\epsilon^2 \mu^2}$</td>
<td>$\left(\frac{1}{\epsilon \mu}\right)^{1/(\epsilon \mu^2)}$</td>
</tr>
<tr>
<td>IDPK</td>
<td>$\left(\frac{1}{\epsilon \mu}\right)^{1/\mu}$, $\frac{1}{\epsilon^2}$</td>
<td>$\left(\frac{1}{\epsilon \mu}\right)^{2/\mu}$, $\frac{1}{\epsilon^4}$</td>
</tr>
</tbody>
</table>
Summary

- Trading data for runtime (?)
- There are more examples of the phenomenon ....

Open questions:
- More points on the curve (new algorithms)
- Lower bounds ??? Can you help ?