Introduction to Machine Learning (67577)
Lecture 2

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PAC learning
1. The PAC Learning Framework
2. No Free Lunch and Prior Knowledge
3. PAC Learning of Finite Hypothesis Classes
4. The Fundamental Theorem of Learning Theory
   - The VC dimension
5. Solving ERM for Halfspaces
Recall: The Game Board

- **Domain set, \( \mathcal{X} \):** This is the set of objects that we may wish to label.
- **Label set, \( \mathcal{Y} \):** The set of possible labels.
- **A prediction rule, \( h : \mathcal{X} \to \mathcal{Y} \):** used to label future examples. This function is called a *predictor*, a *hypothesis*, or a *classifier*.

**Example**

- \( \mathcal{X} = \mathbb{R}^2 \) representing color and shape of papayas.
- \( \mathcal{Y} = \{ \pm 1 \} \) representing “tasty” or “non-tasty”.
- \( h(x) = 1 \) if \( x \) is within the inner rectangle.
Batch Learning

- The learner's input:
  - Training data, $S = ((x_1, y_1) \ldots (x_m, y_m)) \in (\mathcal{X} \times \mathcal{Y})^m$

- The learner's output:
  - A prediction rule, $h : \mathcal{X} \rightarrow \mathcal{Y}$

Intuitively, $h$ should be correct on future examples.
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  - A prediction rule, $h : X \rightarrow Y$

- What should be the goal of the learner?
- Intuitively, $h$ should be correct on future examples
“Correct on future examples”

- Let $f$ be the correct classifier, then we should find $h$ s.t. $h \approx f$
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One way: define the error of $h$ w.r.t. $f$ to be

$$L_{\mathcal{D},f}(h) = \mathbb{P}_{x \sim \mathcal{D}}[h(x) \neq f(x)]$$

where $\mathcal{D}$ is some (unknown) probability measure over $\mathcal{X}$
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where \( \mathcal{D} \) is some (unknown) probability measure over \( \mathcal{X} \)

More formally, \( \mathcal{D} \) is a distribution over \( \mathcal{X} \), that is, for a given \( A \subset \mathcal{X} \), the value of \( \mathcal{D}(A) \) is the probability to see some \( x \in A \). Then,

\[
L_{\mathcal{D}, f}(h) \overset{\text{def}}{=} \mathbb{P}_{x \sim \mathcal{D}}[h(x) \neq f(x)] \overset{\text{def}}{=} \mathcal{D}(\{x \in \mathcal{X} : h(x) \neq f(x)\})
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One way: define the error of $h$ w.r.t. $f$ to be

$$L_{D,f}(h) = \mathbb{P}_{x \sim D} [h(x) \neq f(x)]$$

where $D$ is some (unknown) probability measure over $\mathcal{X}$

More formally, $D$ is a distribution over $\mathcal{X}$, that is, for a given $A \subset \mathcal{X}$, the value of $D(A)$ is the probability to see some $x \in A$. Then,

$$L_{D,f}(h) \overset{\text{def}}{=} \mathbb{P}_{x \sim D} [h(x) \neq f(x)] \overset{\text{def}}{=} D (\{x \in \mathcal{X} : h(x) \neq f(x)\}).$$

Can we find $h$ s.t. $L_{D,f}(h)$ is small?
Data-generation Model

- We must assume some relation between the training data and \( \mathcal{D}, f \)
- Simple data generation model:
  - **Independently Identically Distributed (i.i.d.):** Each \( x_i \) is sampled independently according to \( \mathcal{D} \)
  - **Realizability:** For every \( i \in [m] \), \( y_i = f(x_i) \)
Claim: We can’t hope to find $h$ s.t. $L_{(D,f)}(h) = 0$

Proof: for every $\epsilon \in (0, 1)$ take $X = \{x_1, x_2\}$ and $D(\{x_1\}) = 1 - \epsilon$, $D(\{x_2\}) = \epsilon$

The probability not to see $x_2$ at all among $m$ i.i.d. examples is $(1 - \epsilon)m \approx e^{-\epsilon m}$

So, if $\epsilon \ll 1/m$ we’re likely not to see $x_2$ at all, but then we can’t know its label

Relaxation: We’d be happy with $L_{(D,f)}(h) \leq \epsilon$, where $\epsilon$ is user-specified
Claim: We can’t hope to find $h$ s.t. $L(D, f)(h) = 0$

Proof: for every $\epsilon \in (0, 1)$ take $\mathcal{X} = \{x_1, x_2\}$ and $D(\{x_1\}) = 1 - \epsilon$, $D(\{x_2\}) = \epsilon$
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- So, if $\epsilon \ll 1/m$ we’re likely not to see $x_2$ at all, but then we can’t know its label
- **Relaxation:** We’d be happy with $L_{(\mathcal{D},f)}(h) \leq \epsilon$, where $\epsilon$ is user-specified
Recall that the input to the learner is randomly generated.
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- **Claim:** No algorithm can guarantee $L(D,f)(h) \leq \epsilon$ for sure
- **Relaxation:** We’d allow the algorithm to fail with probability $\delta$, where $\delta \in (0, 1)$ is user-specified
  Here, the probability is over the random choice of examples
Probably Approximately Correct (PAC) learning

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- Learner should output a hypothesis $h$ s.t. with probability of at least $1 - \delta$ it holds that $L_{\mathcal{D},f}(h) \leq \epsilon$
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- Learner should output a hypothesis \( h \) s.t. with probability of at least \( 1 - \delta \) it holds that \( L_{D,f}(h) \leq \epsilon \).
- That is, the learner should be Probably (with probability at least \( 1 - \delta \)) Approximately (up to accuracy \( \epsilon \)) Correct.
No Free Lunch

- Suppose that $|\mathcal{X}| = \infty$
- For any finite $C \subset \mathcal{X}$ take $\mathcal{D}$ to be uniform distribution over $C$
- If number of training examples is $m \leq |C|/2$ the learner has no knowledge on at least half the elements in $C$
- Formalizing the above, it can be shown that:

\[ L_{\mathcal{D},f}(A(S)) \geq \epsilon. \]

Remark: $L_{\mathcal{D},f}(\text{random guess}) = 1/2$, so the theorem states that you can't be better than a random guess.
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**Theorem (No Free Lunch)**

Fix $\delta \in (0, 1), \epsilon < 1/2$. For every learner $A$ and training set size $m$, there exists $\mathcal{D}, f$ such that with probability of at least $\delta$ over the generation of a training data, $S$, of $m$ examples, it holds that $L_{\mathcal{D}, f}(A(S)) \geq \epsilon$. 

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Prior Knowledge

- Give more knowledge to the learner: the target $f$ comes from some hypothesis class, $\mathcal{H} \subset \mathcal{Y}^\mathcal{X}$
- The learner knows $\mathcal{H}$
- Is it possible to PAC learn now?
- Of course, the answer depends on $\mathcal{H}$ since the No Free Lunch theorem tells us that for $\mathcal{H} = \mathcal{Y}^\mathcal{X}$ the problem is not solvable...
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Learning Finite Classes

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  - E.g.: $\mathcal{H}$ is all the functions from $\mathcal{X}$ to $\mathcal{Y}$ that can be implemented using a Python program of length at most $b$
Learning Finite Classes

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  - Input: $\mathcal{H}$ and $S = (x_1, y_1), \ldots, (x_m, y_m)$
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  - Output: any $h \in \mathcal{H}$ s.t. $\forall i, y_i = h(x_i)$

This is also called Empirical Risk Minimization (ERM)

$\text{ERM}_{\mathcal{H}}(S)$

Input: training set $S = (x_1, y_1), \ldots, (x_m, y_m)$

Define the empirical risk:

$L_S(h) = \frac{1}{m} |\{i : h(x_i) \neq y_i\}|$

Output: any $h \in \mathcal{H}$ that minimizes $L_S(h)$
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**ERM$_{\mathcal{H}}(S)$**

- Input: training set $S = (x_1, y_1), \ldots, (x_m, y_m)$
- Define the empirical risk: $L_S(h) = \frac{1}{m} \left| \{i : h(x_i) \neq y_i\} \right|$ 
- Output: any $h \in \mathcal{H}$ that minimizes $L_S(h)$
Fix $\epsilon, \delta$. If $m \geq \frac{\log(|H|/\delta)}{\epsilon}$ then for every $D, f$, with probability of at least $1 - \delta$ (over the choice of $S$ of size $m$), $L_{D,f}(\text{ERM}_H(S)) \leq \epsilon$. 
Proof

- Let $S|_{x} = (x_1, \ldots, x_m)$ be the instances of the training set
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- We would like to prove:

$$D^m \{ S|_x : L(D,f)(\text{ERM}_H(S)) > \epsilon \} \leq \delta$$
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- Let $\mathcal{H}_B$ be the set of “bad” hypotheses,
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- Let $M$ be the set of “misleading” samples,

$$M = \{S|_x : \exists h \in \mathcal{H}_B, L_S(h) = 0\}$$
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- Let $M$ be the set of “misleading” samples,
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- Observe:
  \[ \{ S|_x : L_{(\mathcal{D},f)}(\text{ERM}_{\mathcal{H}}(S)) > \epsilon\} \subseteq M = \bigcup_{h \in \mathcal{H}_B} \{ S|_x : L_S(h) = 0 \} \]
Proof (Cont.)

Lemma (Union bound)

For any two sets $A, B$ and a distribution $D$ we have

$$D(A \cup B) \leq D(A) + D(B).$$
Proof (Cont.)

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- We have shown:
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- Therefore, using the union bound

  $$\mathcal{D}^m(\{S|_x : L_{(\mathcal{D}, f)}(\text{ERM}_\mathcal{H}(S)) > \epsilon \})$$
  $$\leq \sum_{h \in \mathcal{H}_B} \mathcal{D}^m(\{S|_x : L_S(h) = 0\})$$
  $$\leq |\mathcal{H}_B| \max_{h \in \mathcal{H}_B} \mathcal{D}^m(\{S|_x : L_S(h) = 0\})$$
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Observe:

\[ D^m(\{ S \mid x : L_S(h) = 0 \}) = (1 - L_{D,f}(h))^m \]
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- Finally, using \( 1 - \epsilon \leq e^{-\epsilon} \) and \( |\mathcal{H}_B| \leq |\mathcal{H}| \) we conclude:

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- The right-hand side would be \( \leq \delta \) if \( m \geq \frac{\log(|\mathcal{H}|/\delta)}{\epsilon} \). \( \square \)
Illustrating the use of the union bound

Each point is a possible sample $S'_x$. Each colored oval represents misleading samples for some $h \in \mathcal{H}_B$. The probability mass of each such oval is at most $(1 - \epsilon)^m$. But, the algorithm might err if it samples $S'_x$ from any of these ovals.
Outline

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Definition (PAC learnability)

A hypothesis class $\mathcal{H}$ is PAC learnable if there exists a function $m_{\mathcal{H}} : (0, 1)^2 \rightarrow \mathbb{N}$ and a learning algorithm with the following property:

- for every $\epsilon, \delta \in (0, 1)$
- for every distribution $\mathcal{D}$ over $\mathcal{X}$, and for every labeling function $f : \mathcal{X} \rightarrow \{0, 1\}$

when running the learning algorithm on $m \geq m_{\mathcal{H}}(\epsilon, \delta)$ i.i.d. examples generated by $\mathcal{D}$ and labeled by $f$, the algorithm returns a hypothesis $h$ such that, with probability of at least $1 - \delta$ (over the choice of the examples), $L_{(\mathcal{D}, f)}(h) \leq \epsilon$. 
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when running the learning algorithm on $m \geq m_\mathcal{H}(\epsilon, \delta)$ i.i.d. examples generated by $D$ and labeled by $f$, the algorithm returns a hypothesis $h$ such that, with probability of at least $1 - \delta$ (over the choice of the examples), $L(D, f)(h) \leq \epsilon$.

$m_\mathcal{H}$ is called the sample complexity of learning $\mathcal{H}$.
Leslie Valiant, Turing award 2010

For transformative contributions to the theory of computation, including the theory of probably approximately correct (PAC) learning, the complexity of enumeration and of algebraic computation, and the theory of parallel and distributed computing.
What is learnable and how to learn?

- We have shown:

**Corollary**

Let $\mathcal{H}$ be a finite hypothesis class.

- $\mathcal{H}$ is PAC learnable with sample complexity $m_\mathcal{H}(\epsilon, \delta) \leq \frac{\log(|\mathcal{H}|/\delta)}{\epsilon}$
- This sample complexity is obtained by using the $\text{ERM}_\mathcal{H}$ learning rule
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- This sample complexity is obtained by using the ERM$_{\mathcal{H}}$ learning rule

- What about infinite hypothesis classes?
- What is the sample complexity of a given class?
- Is there a generic learning algorithm that achieves the optimal sample complexity?
What is learnable and how to learn?

The fundamental theorem of statistical learning:
- The sample complexity is characterized by the VC dimension
- The ERM learning rule is a generic (near) optimal learner

Chervonenkis

Vapnik
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The VC dimension — Motivation

if someone can explain every phenomena, her explanations are worthless.

Example: http://www.youtube.com/watch?v=p_MzP2MZa0o
Pay attention to the retrospect explanations at 5:00
The VC dimension — Motivation

• Suppose we got a training set \( S = (x_1, y_1), \ldots, (x_m, y_m) \)
The VC dimension — Motivation

- Suppose we got a training set \( S = (x_1, y_1), \ldots, (x_m, y_m) \)
- We try to explain the labels using a hypothesis from \( \mathcal{H} \)
The VC dimension — Motivation

- Suppose we got a training set \( S = (x_1, y_1), \ldots, (x_m, y_m) \)
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- We again try to explain the labels using a hypothesis from \( \mathcal{H} \)
- If this works for us, no matter what the labels are, then something is fishy ...
- Formally, if \( \mathcal{H} \) allows all functions over some set \( C \) of size \( m \), then based on the No Free Lunch, we can’t learn from, say, \( m/2 \) examples
The VC dimension — Formal Definition

- Let $C = \{x_1, \ldots, x_{|C|}\} \subset \mathcal{X}$
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- Let $\mathcal{H}_C$ be the restriction of $\mathcal{H}$ to $C$, namely, $\mathcal{H}_C = \{h_C : h \in \mathcal{H}\}$ where $h_C : C \rightarrow \{0, 1\}$ is s.t. $h_C(x_i) = h(x_i)$ for every $x_i \in C$
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- $\text{VCdim}(\mathcal{H}) = \sup\{|C| : \mathcal{H} \text{ shatters } C\}$
The VC dimension — Formal Definition

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- We say that $\mathcal{H}$ shatters $C$ if $|\mathcal{H}_C| = 2^{|C|}$
- $\text{VCdim}(\mathcal{H}) = \sup\{|C| : \mathcal{H} \text{ shatters } C\}$
- That is, the VC dimension is the maximal size of a set $C$ such that $\mathcal{H}$ gives no prior knowledge w.r.t. $C$
To show that $\text{VCdim}(\mathcal{H}) = d$ we need to show that:

1. There exists a set $C$ of size $d$ which is shattered by $\mathcal{H}$. 
To show that \( \text{VCdim}(\mathcal{H}) = d \) we need to show that:

1. There exists a set \( C \) of size \( d \) which is shattered by \( \mathcal{H} \).
2. Every set \( C' \) of size \( d + 1 \) is not shattered by \( \mathcal{H} \).
Threshold functions: $\mathcal{X} = \mathbb{R}$, $\mathcal{H} = \{ x \mapsto \text{sign}(x - \theta) : \theta \in \mathbb{R} \}$

- Show that $\{0\}$ is shattered
VC dimension — Examples

Threshold functions: \( \mathcal{X} = \mathbb{R}, \mathcal{H} = \{ x \mapsto \text{sign}(x - \theta) : \theta \in \mathbb{R} \} \)

- Show that \( \{0\} \) is shattered
- Show that any two points cannot be shattered
VC dimension — Examples

Intervals: $\mathcal{X} = \mathbb{R}$, $\mathcal{H} = \{h_{a,b} : a < b \in \mathbb{R}\}$, where $h_{a,b}(x) = 1$ iff $x \in [a, b]$

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- Show that $\{0, 1\}$ is shattered
- Show that any three points cannot be shattered
**VC dimension — Examples**

**Axis aligned rectangles:** \( \mathcal{X} = \mathbb{R}^2 \),
\[ \mathcal{H} = \{ h_{(a_1, a_2, b_1, b_2)} : a_1 < a_2 \text{ and } b_1 < b_2 \} \text{, where } h_{(a_1, a_2, b_1, b_2)}(x_1, x_2) = 1 \text{ iff } x_1 \in [a_1, a_2] \text{ and } x_2 \in [b_1, b_2] \]

**Show:**

- **Shattered**
  - \( c_1 \)
  - \( c_4 \)
  - \( c_3 \)

- **Not Shattered**
  - \( c_2 \)
  - \( c_5 \)
VC dimension — Examples

Finite classes:

- Show that the VC dimension of a finite $\mathcal{H}$ is at most $\log_2(|\mathcal{H}|)$. 
VC dimension — Examples

Finite classes:

- Show that the VC dimension of a finite $\mathcal{H}$ is at most $\log_2(|\mathcal{H}|)$.
- Show that there can be arbitrary gap between $VCdim(\mathcal{H})$ and $\log_2(|\mathcal{H}|)$.
VC dimension — Examples

Halfspaces: \( \mathcal{X} = \mathbb{R}^d, \mathcal{H} = \{ x \mapsto \text{sign}(\langle w, x \rangle) : w \in \mathbb{R}^d \} \)

- Show that \( \{ e_1, \ldots, e_d \} \) is shattered
VC dimension — Examples

Halfspaces: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{H} = \{x \mapsto \text{sign}(\langle w, x \rangle) : w \in \mathbb{R}^d\}$

- Show that $\{e_1, \ldots, e_d\}$ is shattered
- Show that any $d + 1$ points cannot be shattered
The Fundamental Theorem of Statistical Learning

Theorem (The Fundamental Theorem of Statistical Learning)

Let $\mathcal{H}$ be a hypothesis class of binary classifiers. Then, there are absolute constants $C_1, C_2$ such that the sample complexity of PAC learning $\mathcal{H}$ is

$$C_1 \frac{d + \log(1/\delta)}{\epsilon} \leq m_\mathcal{H}(\epsilon, \delta) \leq C_2 \frac{d \log(1/\epsilon) + \log(1/\delta)}{\epsilon}$$

Furthermore, this sample complexity is achieved by the ERM learning rule.
Proof of the lower bound – main ideas

- Suppose $\text{VCdim}(\mathcal{H}) = d$ and let $C = \{x_1, \ldots, x_d\}$ be a shattered set.
Proof of the lower bound – main ideas

- Suppose $\text{VCdim}(\mathcal{H}) = d$ and let $C = \{x_1, \ldots, x_d\}$ be a shattered set.
- Consider the distribution $\mathcal{D}$ supported on $C$ s.t.

$$\mathcal{D}(\{x_i\}) = \begin{cases} 1 - 4\epsilon & \text{if } i = 1 \\ 4\epsilon/(d - 1) & \text{if } i > 1 \end{cases}$$
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  \[ D(\{x_i\}) = \begin{cases} 1 - 4\epsilon & \text{if } i = 1 \\ 4\epsilon/(d-1) & \text{if } i > 1 \end{cases} \]

- If we see $m$ i.i.d. examples then the expected number of examples from $C \setminus \{x_1\}$ is $4\epsilon m$. 

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Proof of the lower bound – main ideas

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  \end{cases}
  \]
- If we see $m$ i.i.d. examples then the expected number of examples from $C \setminus \{x_1\}$ is $4\epsilon m$.
- If $m < \frac{d-1}{8\epsilon}$ then $4\epsilon m < \frac{d-1}{2}$ and therefore, we have no information on the labels of at least half the examples in $C \setminus \{x_1\}$. 
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- If $m < \frac{d-1}{8\epsilon}$ then $4\epsilon m < \frac{d-1}{2}$ and therefore, we have no information on the labels of at least half the examples in $C \setminus \{x_1\}$.
- Best we can do is to guess, but then our error is $\geq \frac{1}{2} \cdot 2\epsilon = \epsilon$. 
Proof of the upper bound – main ideas

- Recall our proof for finite class:

For a single hypothesis, we've shown that the probability of the event:

\[ L_S(h) = 0 \]

given that \( L(D, f) > \epsilon \) is at most

\[ e^{-\epsilon m} \]

Then we applied the union bound over all “bad” hypotheses, to obtain the bound on ERM failure:

\[ |H| e^{-\epsilon m} \]

If \( H \) is infinite, or very large, the union bound yields a meaningless bound.
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- **The two samples trick:** show that

\[
P_{S \sim D^m} \left[ \exists h \in \mathcal{H}_B : L_S(h) = 0 \right] \\
\leq 2 P_{S,T \sim D^m} \left[ \exists h \in \mathcal{H}_B : L_S(h) = 0 \text{ and } L_T(h) \geq \epsilon/2 \right]
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- **Symmetrization:** Since \( S, T \) are i.i.d., we can think on first sampling \( 2m \) examples and then splitting them to \( S, T \) at random.
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- If we fix \( h \), and \( S \cup T \), the probability to have \( L_S(h) = 0 \) while \( L_T(h) \geq \epsilon/2 \) is \( \leq e^{-em/4} \)
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- If we fix \( h \), and \( S \cup T \), the probability to have \( L_S(h) = 0 \) while \( L_T(h) \geq \epsilon/2 \) is \( \leq e^{-\epsilon m/4} \)

- Once we fixed \( S \cup T \), we can take a union bound only over \( \mathcal{H}_{S \cup T} \)
Proof of the upper bound – main ideas

Lemma (Sauer-Shelah-Perles$^2$-Vapnik-Chervonenkis)

Let $\mathcal{H}$ be a hypothesis class with $\text{VCdim}(\mathcal{H}) \leq d < \infty$. Then, for all $C \subset \mathcal{X}$ s.t. $|C| = m > d + 1$ we have

$$|\mathcal{H}_C| \leq \left(\frac{em}{d}\right)^d$$
Outline

1. The PAC Learning Framework
2. No Free Lunch and Prior Knowledge
3. PAC Learning of Finite Hypothesis Classes
4. The Fundamental Theorem of Learning Theory
   - The VC dimension
5. Solving ERM for Halfspaces
ERM for halfspaces

- Recall:
  \[ \mathcal{H} = \{ x \mapsto \text{sign}(\langle w, x \rangle) : w \in \mathbb{R}^d \} \]
ERM for halfspaces

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  \[ \mathcal{H} = \{ x \mapsto \text{sign}(\langle w, x \rangle) : w \in \mathbb{R}^d \} \]

- ERM for Halfspaces:
  \begin{align*}
  \text{given } S = (x_1, y_1), \ldots, (x_m, y_m) \text{ find } w \text{ s.t. for all } i, \\
  \text{sign}(\langle w, x_i \rangle) = y_i.
  \end{align*}
ERM for halfspaces

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- Cast as a Linear Program:
  Find \( w \) s.t.
  \[ \forall i, \quad y_i \langle w, x_i \rangle > 0. \]
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- Can solve efficiently using standard methods
ERM for halfspaces

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- Cast as a Linear Program:
  Find \( \mathbf{w} \) s.t.
  \[ \forall i, \quad y_i \langle \mathbf{w}, \mathbf{x}_i \rangle > 0. \]

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- Exercise: show how to solve the above Linear Program using the Ellipsoid learner from the previous lecture
ERM for halfspaces using the Perceptron Algorithm

Perceptron

initialize:  \( w = (0, \ldots, 0) \in \mathbb{R}^d \)

while \( \exists i \) s.t. \( y_i \langle w, x_i \rangle \leq 0 \)

\( w \leftarrow w + y_i x_i \)

- Dates back at least to Rosenblatt 1958.
Theorem (Agmon’54, Novikoff’62)

Let \((x_1, y_1), \ldots, (x_m, y_m)\) be a sequence of examples such that there exists \(w^* \in \mathbb{R}^d\) such that for every \(i\), \(y_i \langle w^*, x_i \rangle \geq 1\). Then, the Perceptron will make at most

\[\|w^*\|^2 \max_i \|x_i\|^2\]

updates before breaking with an ERM halfspace.
Theorem (Agmon’54, Novikoff’62)

Let \((x_1, y_1), \ldots, (x_m, y_m)\) be a sequence of examples such that there exists \(w^* \in \mathbb{R}^d\) such that for every \(i\), \(y_i \langle w^*, x_i \rangle \geq 1\). Then, the Perceptron will make at most 

\[
\|w^*\|_2^2 \max_i \|x_i\|_2^2
\]

updates before breaking with an ERM halfspace.

- The condition would always hold if the data is realizable by some halfspace
- However, \(\|w^*\|\) might be very large
- In many practical cases, \(\|w^*\|\) would not be too large
Proof

Let $w(t)$ be the value of $w$ at iteration $t$
Proof

- Let $w^{(t)}$ be the value of $w$ at iteration $t$
- Let $(x_t, y_t)$ be the example used to update $w$ at iteration $t$
Proof

- Let $w^{(t)}$ be the value of $w$ at iteration $t$
- Let $(x_t, y_t)$ be the example used to update $w$ at iteration $t$
- Denote $R = \max_i \|x_i\|$

The cosine of the angle between $w^*$ and $w^{(t)}$ is
\[
\langle w^{(t)}, w^* \rangle \leq \|w^{(t)}\| \|w^*\|
\]
By the Cauchy-Schwartz inequality, this is always $\leq 1$

We will show:
\[
1 \geq \langle w^{(t+1)}, w^* \rangle \leq R \sqrt{t}
\]
This would yield:
\[
R \sqrt{t} \|w^*\| \leq \langle w^{(t)}, w^* \rangle \|w^{(t)}\| \|w^*\| \leq 1
\]
Rearranging the above would yield
\[
t \leq \|w^*\|^2 R^2
\]
as required.
Proof

- Let $w^{(t)}$ be the value of $w$ at iteration $t$
- Let $(x_t, y_t)$ be the example used to update $w$ at iteration $t$
- Denote $R = \max_i \|x_i\|$
- The cosine of the angle between $w^*$ and $w^{(t)}$ is $\frac{\langle w^{(t)}, w^* \rangle}{\|w^{(t)}\| \|w^*\|}$
Proof

- Let \( w^{(t)} \) be the value of \( w \) at iteration \( t \)
- Let \((x_t, y_t)\) be the example used to update \( w \) at iteration \( t \)
- Denote \( R = \max_i \|x_i\| \)
- The cosine of the angle between \( w^* \) and \( w^{(t)} \) is \( \frac{\langle w^{(t)}, w^* \rangle}{\|w^{(t)}\| \|w^*\|} \)
- By the Cauchy-Schwartz inequality, this is always \( \leq 1 \)
Proof

- Let \( w(t) \) be the value of \( w \) at iteration \( t \)
- Let \((x_t, y_t)\) be the example used to update \( w \) at iteration \( t \)
- Denote \( R = \max_i ||x_i|| \)

The cosine of the angle between \( w^* \) and \( w(t) \) is
\[
\frac{\langle w(t), w^* \rangle}{||w(t)|| \cdot ||w^*||}
\]

By the Cauchy-Schwartz inequality, this is always \( \leq 1 \)

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  1. $\langle w^{(t+1)}, w^* \rangle \geq t$

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  2. \( \|w^{(t+1)}\| \leq R \sqrt{t} \)
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- Denote $R = \max_i \|x_i\|$
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- We will show:
  1. $\langle w^{(t+1)}, w^* \rangle \geq t$
  2. $\|w^{(t+1)}\| \leq R \sqrt{t}$
- This would yield

$$\frac{t}{R \sqrt{t} \|w^*\|} \leq \frac{\langle w^{(t)}, w^* \rangle}{\|w^{(t)}\| \|w^*\|} \leq 1$$
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- This would yield

$$\frac{t}{R \sqrt{t} \|w^*\|} \leq \frac{\langle w^{(t)}, w^* \rangle}{\|w^{(t)}\| \|w^*\|} \leq 1$$

- Rearranging the above would yield $t \leq \|w^*\|^2 R^2$ as required.
Proof (Cont.)

Showing $\langle \mathbf{w}^{(t+1)}, \mathbf{w}^* \rangle \geq t$

- Initially, $\langle \mathbf{w}^{(1)}, \mathbf{w}^* \rangle = 0$

Showing $\|\mathbf{w}^{(t+1)}\|_2^2 \leq R^2 t$
Proof (Cont.)

Showing \( \langle \mathbf{w}^{(t+1)}, \mathbf{w}^* \rangle \geq t \)
- Initially, \( \langle \mathbf{w}^{(1)}, \mathbf{w}^* \rangle = 0 \)
- Whenever we update, \( \langle \mathbf{w}^{(t)}, \mathbf{w}^* \rangle \) increases by at least 1:
  \[
  \langle \mathbf{w}^{(t+1)}, \mathbf{w}^* \rangle = \langle \mathbf{w}^{(t)} + yt \mathbf{x}_t, \mathbf{w}^* \rangle = \langle \mathbf{w}^{(t)}, \mathbf{w}^* \rangle + \underbrace{yt \langle \mathbf{x}_t, \mathbf{w}^* \rangle}_{\geq 1} 
  \]

Showing \( \| \mathbf{w}^{(t+1)} \|^2 \leq R^2 t \)
Proof (Cont.)

Showing \( \langle w^{(t+1)}, w^* \rangle \geq t \)
- Initially, \( \langle w^{(1)}, w^* \rangle = 0 \)
- Whenever we update, \( \langle w^{(t)}, w^* \rangle \) increases by at least 1:
  \[
  \langle w^{(t+1)}, w^* \rangle = \langle w^{(t)} + y_t x_t, w^* \rangle = \langle w^{(t)}, w^* \rangle + y_t \langle x_t, w^* \rangle \geq 1
  \]

Showing \( \| w^{(t+1)} \|^2 \leq R^2 t \)
- Initially, \( \| w^{(1)} \|^2 = 0 \)
Proof (Cont.)

Showing $\langle w^{(t+1)}, w^* \rangle \geq t$

- Initially, $\langle w^{(1)}, w^* \rangle = 0$
- Whenever we update, $\langle w^{(t)}, w^* \rangle$ increases by at least 1:

$$\langle w^{(t+1)}, w^* \rangle = \langle w^{(t)} + yt x_t, w^* \rangle = \langle w^{(t)}, w^* \rangle + yt \langle x_t, w^* \rangle \geq 1$$

Showing $\|w^{(t+1)}\|^2 \leq R^2 t$

- Initially, $\|w^{(1)}\|^2 = 0$
- Whenever we update, $\|w^{(t)}\|^2$ increases by at most 1:

$$\|w^{(t+1)}\|^2 = \|w^{(t)} + yt x_t\|^2 = \|w^{(t)}\|^2 + 2yt \langle w^{(t)}, x_t \rangle + y_t^2 \|x_t\|^2 \leq 0$$

$$\leq \|w^{(t)}\|^2 + R^2 .$$
Summary

- The PAC Learning model
- What is PAC learnable?
- PAC learning of finite classes using ERM
- The VC dimension and the fundamental theorem of learning
  - Classes of finite VC dimension
- How to PAC learn?
  - Using ERM
- Learning halfspaces using: Linear programming, Ellipsoid, Perceptron