

COUNTING MAGIC SQUARES IN QUASI-POLYNOMIAL TIME

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ABSTRACT. We present a randomized algorithm, which, given positive integers n and t and a real number $0 < \epsilon < 1$, computes the number $|\Sigma(n, t)|$ of $n \times n$ non-negative integer matrices (magic squares) with the row and column sums equal to t within relative error ϵ . The computational complexity of the algorithm is polynomial in ϵ^{-1} and quasi-polynomial in $N = nt$, that is, of the order $N^{\log N}$. A simplified version of the algorithm works in time polynomial in ϵ^{-1} and N and estimates $|\Sigma(n, t)|$ within a factor of $N^{\log N}$. This simplified version has been implemented. We present results of the implementation, state some conjectures, and discuss possible generalizations.

1. INTRODUCTION AND MAIN RESULTS

(1.1) Magic squares. Let us fix two positive integers n and t . An $n \times n$ *magic square with the line sum t* is an $n \times n$ non-negative integer matrix $D = (d_{ij})$ with the row and column sums t :

$$\sum_{j=1}^n d_{ij} = t \quad \text{for } i = 1, \dots, n \quad \text{and}$$
$$\sum_{i=1}^n d_{ij} = t \quad \text{for } j = 1, \dots, n.$$

We note that sometimes such matrices are called semi-magic squares, but we follow the terminology adopted in modern combinatorics, for example, in Chapter 4 of [St97].

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Let $\Sigma(n, t)$ be the set of all $n \times n$ magic squares with the line sum t . In this paper, we present a randomized approximation algorithm to compute the number $|\Sigma(n, t)|$. The algorithm runs in quasi-polynomial time. More precisely, let $N = nt$ (in what follows, we reserve notation N for the sum of the entries of the matrix).

We present a randomized algorithm, which, for any given positive integers n and t and positive $\epsilon < 1$, approximates $|\Sigma(n, t)|$ within relative error ϵ . The computational complexity of the algorithm is $(1/\epsilon)^{O(1)} N^{O(\ln N)}$ (in the unit cost model).

From this same approach, one also obtains a simpler, randomized polynomial time algorithm which approximates $|\Sigma(n, t)|$ within a factor of $N^{O(\ln N)}$. We implemented the latter algorithm and report on the computational results in Section 1.4.

(1.2) Contingency tables. More generally, given positive integers m and n , a positive integer vector $R = (r_1, \dots, r_m)$, and a positive integer vector $C = (c_1, \dots, c_n)$ such that

$$r_1 + \dots + r_m = c_1 + \dots + c_n = N,$$

the $m \times n$ non-negative integer matrices with the row sums r_1, \dots, r_m and the column sums c_1, \dots, c_n are called *contingency tables* with the margins R and C . The problem of computing or estimating efficiently the cardinality $|\Sigma(R, C)|$ of the set of contingency tables with the given margins has been of significant interest, see [DE85], [DG95], [D+97], [Mo02], and [CD03] due to connections to statistics, representation theory, and symmetric functions.

Using the Markov Chain Monte Carlo approach, Dyer, Kannan, and Mount [D+97] showed how to count contingency tables if the row and column sums are sufficiently large, namely, if $r_i = \Omega(n^2 m)$ and $c_j = \Omega(m^2 n)$ for all i, j . They presented a randomized algorithm, which, given an $\epsilon > 0$, approximates the number $|\Sigma(R, C)|$ of tables within relative error ϵ in time polynomial in ϵ^{-1} , n , m , and $\sum_i \log r_i + \sum_j \log c_j$ (the bit size of the margins). It turns out that for large margins the number of contingency tables is well-approximated by the volume of the *transportation polytope* of the $m \times n$ non-negative matrices with the row sums r_i and the column sums c_j . The set $\Sigma(R, C)$ of contingency tables can be viewed as the set of integer points in that polytope. Subsequently, Morris [Mo02] obtained a similar result for the bounds $r_i = \Omega(n^{2/3} m \ln m)$ and $c_j = \Omega(m^{3/2} n \ln n)$.

In addition, for large r_i and c_j there is a heuristic formula for $|\Sigma(R, C)|$ due to Diaconis and Efron [DE85]:

$$(1.2.1) \quad |\Sigma(R, C)| \approx \left(\frac{2N + mn}{2} \right)^{(m-1)(n-1)} \left(\prod_{i=1}^m \bar{r}_i \right)^{n-1} \\ \times \left(\prod_{j=1}^n \bar{c}_j \right)^{m-1} \frac{\Gamma(nk)}{\Gamma^m(n) \Gamma^n(k)},$$

where

$$\begin{aligned} \bar{r}_i &= \frac{1-w}{m} + \frac{wr_i}{N} \quad \text{and} \quad c_j = \frac{1-w}{n} + \frac{wc_j}{N} \\ \text{for } w &= \frac{1}{1+mn/2N} \quad \text{and} \quad k = \frac{n+1}{n \sum_{i=1}^m \bar{r}_i^2} - \frac{1}{n}. \end{aligned}$$

This formula first approximates $|\Sigma(R, C)|$ by the volume of the corresponding transportation polytope and then (since no explicit formula is known for the volume) approximates the volume by an explicitly computable integral of a certain density. However, there are no proved or even conjectural conclusions on the accuracy of this formula.

At the opposite extreme, when the margins r_i, c_j are very small (bounded by a constant fixed in advance) relative to the dimensions m and n of the matrix, Békéssy, Békéssy, and Komlós [B+72] obtained an asymptotic formula

$$(1.2.2) \quad |\Sigma(R, C)| \approx \frac{N!}{r_1! \cdots r_m! c_1! \cdots c_n!} \exp \left\{ \frac{2}{N^2} \sum_{i,j} \binom{r_i}{2} \binom{c_j}{2} \right\}.$$

This formula reflects the fact that the majority of contingency tables with small margins have entries 0, 1, and 2. Also, if the margins are bounded by a constant fixed in advance, one can compute the exact value of $|\Sigma(R, C)|$ in time polynomial in $m+n$ by a dynamic programming algorithm.

Using the dynamic programming approach in a different vein, Cryan and Dyer [CD03] constructed a randomized polynomial time approximation algorithm for computing $|\Sigma(R, C)|$ provided the number m of rows (or the number n of columns) is fixed in advance.

In some sense, the case of magic squares $m=n$ with moderately large margins t (say, of order n) lies at the core of the remaining hard cases of contingency table enumeration. Our algorithm is quasi-polynomial, and we conjecture that its straightforward modification achieves, in fact, a genuinely polynomial time complexity, and that it naturally extends to a randomized polynomial time algorithm to count contingency tables with any margins, see Section 10. A disadvantage of our approach is that there seems to be no easy way to generate a *random* magic square, unlike in the approaches of [D+97] and [Mo02].

(1.3) Idea of the algorithm. Our algorithm builds on the technique of rapidly mixing Markov chains and, in particular on efficient sampling from log-concave densities, as developed in [AK91], [F+94], [FK99], [LV06], see also [Ve05] for a survey, the permanent approximation algorithm [J+04], the strongly polynomial time algorithm for matrix scaling [L+00], as well as the integral representation of $|\Sigma(R, C)|$ from [Ba05] and [Ba07].

Let $\Delta = \Delta_{n \times n} \subset \mathbb{R}^{n^2}$ be the open $(n^2 - 1)$ -dimensional simplex of all $n \times n$ positive matrices $X = (x_{ij})$ such that

$$\sum_{i,j=1}^n x_{ij} = 1.$$

Let μ be Lebesgue measure on Δ normalized by the constraint $\mu(\Delta) = 1$. Using results of [Ba05], we represent the number of magic squares as an integral

$$(1.3.1) \quad |\Sigma(n, t)| = \int_{\Delta} f \, d\mu$$

of some continuous density $f : \Delta \rightarrow \mathbb{R}_+$. Furthermore, as in [Ba05], the density f is factored

$$(1.3.2) \quad f = p\phi,$$

where $\phi : \Delta \rightarrow \mathbb{R}_+$ is *log-concave*, that is,

$$\begin{aligned} \phi(\alpha X + \beta Y) &\geq \phi^\alpha(X)\phi^\beta(Y) \quad \text{for all } X, Y \in \Delta \quad \text{and} \\ &\text{for all } \alpha, \beta \geq 0 \quad \text{such that } \alpha + \beta = 1 \end{aligned}$$

and $p(X) \geq 1$ for all $X \in \Delta$. Moreover, for any $X \in \Delta$ the values $p(X)$ and $\phi(X)$ are computable in time polynomial in N . More precisely, for any given $\epsilon > 0$ the value of ϕ can be computed within relative error ϵ in time polynomial in $\ln(1/\epsilon)$ and N by a deterministic algorithm of [L+00] while the value of p can be computed within relative error ϵ in time polynomial in $1/\epsilon$ and N by a randomized algorithm of [J+04]. The algorithm computing ϕ seems to work very well in practice.

The key result of this paper is that there is a *threshold* $T = N^{\kappa \ln N}$ for some absolute constant $\kappa > 0$ such that if we define $\bar{p} : \Delta \rightarrow \mathbb{R}_+$ by

$$\bar{p}(X) = \begin{cases} p(X) & \text{if } p(X) \leq T \\ T & \text{if } p(X) > T \end{cases}$$

then the integral

$$(1.3.3) \quad \int_{\Delta} \bar{p}\phi \, d\mu$$

approximates the integral

$$\int_{\Delta} p\phi \, d\mu = |\Sigma(n, t)|$$

within a relative error as small as N^{-n} , say.

The simplified version of the algorithm consists of computing the integral

$$(1.3.4) \quad \int_{\Delta} \phi \, d\mu$$

using any of the randomized polynomial time algorithms of [AK91], [F+94], [FK99], and [LV06] for *integrating* log-concave densities. Clearly, the value of (1.3.4) approximates $|\Sigma(n, t)|$ within a factor of $N^{O(\ln N)}$, that is,

$$\int_{\Delta} \phi \, d\mu \leq |\Sigma(n, t)| \leq N^{\kappa \ln N} \int_{\Delta} \phi \, d\mu$$

for some absolute constant $\kappa \geq 0$.

The full version of the algorithm consists of estimating (1.3.3) within relative error ϵ using any of the randomized polynomial time algorithms of [AK91], [F+94], [FK99], and [LV06] for *sampling* from log-concave densities as well as the randomized polynomial time algorithm of [J+04] for approximating the permanent of a positive matrix.

Namely, let ν be the Borel probability measure on Δ with the density proportional to ϕ .

Thus we can rewrite (1.3.3) as the product

$$\left(\int_{\Delta} \bar{p} \, d\nu \right) \left(\int_{\Delta} \phi \, d\mu \right).$$

We compute the second factor as above. The first factor is approximated by the sample mean

$$(1.3.5) \quad \int_{\Delta} \bar{p} \, d\nu \approx \frac{1}{m} \sum_{i=1}^m \bar{p}(x_i),$$

where $x_1, \dots, x_m \in \Delta$ are independent points sampled at random from the measure ν . Each such point can be sampled in time polynomial in N . The Chebyshev inequality implies that to achieve relative error ϵ with probability $2/3$ it suffices to sample $m = O(\epsilon^{-2} T^2) = \epsilon^{-2} N^{O(\ln N)}$ points in (1.3.5).

(1.4) Computational experiments.

We implemented the simplified version of the algorithm that computes the integral (1.3.4). Below, we have tabulated some representative examples of our results for various values of t and n . The software and additional data are available at [Yo07].

For $n \leq 7$, we were able to compare the obtained values (“estimate” in the table below) against the exact numbers (“answer” in the table below is the exact number rounded to the three main digits) computed at our request by Jesús De Loera using the `LattE` code, see [L+04]:

n	t	estimate	answer
5	5	1.51×10^7	2.21×10^7
	10	7.96×10^{10}	7.93×10^{10}
	125	1.55×10^{27}	1.10×10^{27}
6	6	4.70×10^{11}	6.02×10^{11}
	12	7.12×10^{16}	2.28×10^{17}
	36	2.21×10^{27}	5.62×10^{27}
	216	3.76×10^{46}	3.07×10^{46}
7	7	1.70×10^{17}	2.16×10^{17}
	14	1.75×10^{25}	2.46×10^{25}
	49	3.00×10^{42}	4.00×10^{42}
	343	3.90×10^{71}	1.28×10^{72} .

The larger values of n below seem to be presently beyond the reach of `LattE` or any other available computer code. Thus we compared the obtained values (“estimate” in the table below) with the Diaconis-Efron heuristic formula (1.2.1) (“heuristic” in the table below), which is believed to be valid for $t \gg n$:

n	t	estimate	heuristic
12	8	2.72×10^{48}	4.96×10^{49}
	20	4.55×10^{81}	1.68×10^{82}
15	20	8.3×10^{119}	2.43×10^{121}
	100	7.65×10^{236}	2.71×10^{237} .

An interesting feature of the data is that the Diaconis-Efron formula seems to be in a reasonable agreement with our computations even at the range $t \sim n$, where the heuristic arguments supporting the formula do not look plausible any longer. One can argue, however, that the agreement becomes more reasonable for larger t .

Finally, we consider the case of very small t , where we compare our results (“estimate” in the table below) with the asymptotic formula (1.2.2) (“asymptotic” in the table below).

n	t	estimate	asymptotic
25	5	2.89×10^{108}	6.17×10^{108}
30	5	1.49×10^{142}	3.02×10^{142} .

These results suggest that the integral (1.3.4) approximates $|\Sigma(n, t)|$ quite well, conjecturally within a factor of $N^{O(1)}$, if not just $O(1)$. Moreover, the algorithm appears to estimate $|\Sigma(R, C)|$ for general margins (R, C) with similar effectiveness. For example, in a known test case [DE85], [L+04], we have $n = 4$,

$R = (220, 215, 93, 64)$ and $C = (108, 286, 71, 127)$. The correct answer is about 1.22×10^{15} whereas our algorithm predicts it to be 1.02×10^{15} .

In practice, our algorithm is slower than **LattE** when the latter is applicable. However, as is explained in Section 3.6, our algorithm has the advantage of being able to compute when the values of n are moderately large, far beyond those possible with exact methods. Typically, we have been able to compute these in the order of a few days on current technology. However, the algorithm is highly parallelizable, and also, the intermediate output used to estimate $|\Sigma(n, t)|$ can be used to “bootstrap” estimates of $|\Sigma(n, u)|$ for $u > t$. Exploiting these features substantially decrease computing time. Finally, the memory requirements of our implementation are modest and have not been an issue in our experiments.

(1.5) Organization of the paper.

In Section 2, we describe the density f of formula (1.3.1) and the factorization $f = p\phi$ of (1.3.2).

In Section 3, we describe the algorithm of approximating the number $|\Sigma(n, t)|$ of magic squares in detail and state all the bounds for $|\Sigma(n, t)|$, f , ϕ , and p we need to conclude that the algorithm indeed approximates the desired number within relative error ϵ in $(1/\epsilon)^{O(1)} N^{O(\ln N)}$ time. We also describe details of the implementation.

Sections 4-9 are devoted to the proofs.

In Section 4, we invoke certain classical results, the van der Waerden and the Bregman-Minc bounds for the permanent of a non-negative matrix and obtain a straightforward corollary that we use later.

In Section 5, we prove that the total number $\binom{N+n^2-1}{n^2-1}$ of $n \times n$ non-negative integer matrices with the sum of entries equal to N is at most $N^{O(n)}$ times bigger than the number of $n \times n$ magic squares with the line sum t . Also, we prove that the maximum of the density f on the simplex Δ in (1.3.1) does not exceed $\binom{N+n^2-1}{n^2-1}$.

In Sections 6–8, we prove the key estimate of the paper, namely, that for any $\alpha > 0$ there is a $\beta = \beta(\alpha) > 0$ such that the probability that a random $X \in \Delta$ satisfies $p(X) > N^{\beta \ln N}$ in (1.3.2) does not exceed $N^{-\alpha n}$. Section 7 contains some standard probabilistic estimates whereas Section 6 contains an estimate of the entries of the doubly stochastic scaling of a positive matrix, which may be of interest in its own right. Roughly, it states that for a sufficiently generic $n \times n$ matrix A , all the entries of its doubly stochastic scaling are sufficiently close to $1/n$.

In Section 9 we state some technical estimates for the log-concave density ϕ which imply that the algorithms of [AK91], [F+94], [FK99], [LV06] for polynomial time integration and sampling are indeed applicable.

Finally, in Section 10 we describe possible extensions of our approach, in particular, to contingency tables with equal row sums, but not necessarily column sums and vice versa. We also conjecture that the $N^{O(\ln N)}$ bound can be replaced by $N^{O(1)}$ so that our approach produces a polynomial time algorithm.

2. THE INTEGRAL REPRESENTATION FOR THE NUMBER OF MAGIC SQUARES

To obtain the representation (1.3.1), we express the number $|\Sigma(n, t)|$ as the expectation of the permanent of an $N \times N$ random matrix for $N = nt$. Let $A = (a_{ij})$ be an $N \times N$ square matrix. The *permanent* of A is given by the formula

$$\text{per } A = \sum_{\sigma \in S_N} \prod_{i=1}^N a_{i\sigma(i)},$$

where S_N is the symmetric group of all permutations σ of the set $\{1, \dots, N\}$.

Let J_t denote the $t \times t$ matrix with the entries $1/t$. For an $n \times n$ matrix $X = (x_{ij})$, the matrix $X \otimes J_t$ denotes the $N \times N$ block matrix whose (i, j) th block is the $t \times t$ matrix with the entries x_{ij}/t .

We recall that a random variable ξ has the *standard exponential* distribution if

$$\mathbf{P}\{\xi > \tau\} = \begin{cases} 1 & \text{if } \tau \leq 0 \\ e^{-\tau} & \text{if } \tau > 0. \end{cases}$$

The following result is a particular case of the general formula of Theorem 1.2 of [Ba05] and Theorem 4 of [Ba07] for the number of contingency tables of a given type.

(2.1) Theorem. *Let $X = (x_{ij})$ be the $n \times n$ matrix of independent standard exponential random variables x_{ij} . Then*

$$|\Sigma(n, t)| = \frac{t^N}{(t!)^{2n}} \mathbf{E} \text{ per } (X \otimes J_t).$$

In other words,

$$|\Sigma(n, t)| = \frac{t^N}{(t!)^{2n}} \int_{\mathbb{R}^{n^2}} \text{per } (X \otimes J_t) \exp \left\{ - \sum_{i,j=1}^n x_{ij} \right\} dx,$$

where dx is the standard Lebesgue measure in the space \mathbb{R}^{n^2} , interpreted as the space of $n \times n$ matrices $X = (x_{ij})$.

Let

$$\Delta = \Delta_{n \times n} = \left\{ X = (x_{ij}) : x_{ij} > 0 \text{ for all } i, j \text{ and } \sum_{i,j=1}^n x_{ij} = 1 \right\}$$

be the standard $(n^2 - 1)$ -dimensional (open) simplex in \mathbb{R}^{n^2} endowed with the probability measure μ that is the normalization of the Lebesgue measure on $\Delta_{n \times n}$.

Since $\text{per } (X \otimes J_t)$ is a homogeneous polynomial of degree $N = nt$ in the entries x_{ij} of X , we obtain the following integral representation, see Section 4 of [Ba05].

(2.2) Corollary.

$$|\Sigma(n, t)| = \frac{(N + n^2 - 1)! t^N}{(n^2 - 1)! (t!)^{2n}} \int_{\Delta_{n \times n}} \text{per}(X \otimes J_t) d\mu(X).$$

Thus we define $f : \Delta \rightarrow \mathbb{R}_+$ in representation (1.3.1) by

$$(2.3) \quad f(X) = \frac{(N + n^2 - 1)! t^N}{(n^2 - 1)! (t!)^{2n}} \text{per}(X \otimes J_t) \quad \text{for } X \in \Delta_{n \times n}.$$

(2.4) Factoring the density. Now we describe how to factor the density $f = p\phi$ in (1.3.2), where $\phi : \Delta \rightarrow \mathbb{R}_+$ is a log-concave function and $p : \Delta \rightarrow \mathbb{R}_+$ is a function which “does not vary much” on Δ . We employ the notion of *matrix scaling*, see [Si64], [MO68], [KK96], [L+00].

Let $X = (x_{ij})$ be an $n \times n$ positive matrix. Then there exists an $n \times n$ positive matrix $Y = (y_{ij})$ and positive numbers λ_i, μ_i for $i = 1, \dots, n$ such that

$$x_{ij} = y_{ij} \lambda_i \mu_j \quad \text{for } i, j = 1, \dots, n$$

and Y is *doubly stochastic*, that is,

$$\begin{aligned} \sum_{j=1}^n y_{ij} &= 1 \quad \text{for } i = 1, \dots, n \quad \text{and} \\ \sum_{i=1}^n y_{ij} &= 1 \quad \text{for } j = 1, \dots, n. \end{aligned}$$

Furthermore, given matrix X , the matrix Y is unique (we call it the *doubly stochastic scaling of X*) while the factors λ_i and μ_j are unique up to a rescaling $\lambda_i := \lambda_i \tau, \mu_i := \mu_i \tau^{-1}$ for some $\tau > 0$ and $i = 1, \dots, n$. This allows us to define a function $\sigma : \Delta \rightarrow \mathbb{R}_+$ by

$$\sigma(X) = \prod_{i=1}^n (\lambda_i \mu_i).$$

Clearly, $\text{per } X = (\text{per } Y) \sigma(X)$.

The crucial fact about σ that we use is that σ is log-concave, that is,

$$\sigma(\alpha_1 X_1 + \alpha_2 X_2) \geq \sigma^{\alpha_1}(X_1) \sigma^{\alpha_2}(X_2)$$

for all $X_1, X_2 \in \Delta$ and all $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$, see [GS02], [Gu06], [Ba05], [Ba06]. Also, for any given X , the value of σ can be computed within relative error ϵ in time polynomial in $\ln(1/\epsilon)$ and n [L+00].

One can easily see that if Y is the doubly stochastic scaling of X then $Y \otimes J_t$ is the doubly stochastic scaling of $X \otimes J_t$ and that

$$\sigma(X \otimes J_t) = \sigma^t(X).$$

Moreover,

$$\text{per}(X \otimes J_t) = \text{per}(Y \otimes J_t) \sigma^t(X).$$

We define

$$(2.4.1) \quad \begin{aligned} f &= p\phi \quad \text{where} \\ p(X) &= \frac{N^N}{N!} \text{per}(Y \otimes J_t) \quad \text{and} \\ \phi(X) &= \frac{(N + n^2 - 1)! N! t^N}{(n^2 - 1)! (t!)^{2n} N^N} \sigma^t(X), \end{aligned}$$

for $N = nt$. Clearly, ϕ is log-concave and Corollary 2.2 implies that

$$(2.4.2) \quad |\Sigma(n, t)| = \int_{\Delta} p\phi \, d\mu.$$

We note that for any given $X \in \Delta$ and $\epsilon > 0$ the value of $p(X)$ can be computed within relative error ϵ in time polynomial in N and ϵ^{-1} by the randomized algorithm of [J+04].

It is convenient to define f and ϕ on the set Mat_+ of positive $n \times n$ matrices and not just on the simplex Δ .

3. BOUNDS AND THE DETAILED DESCRIPTION OF THE ALGORITHM

In this section we give the detailed description of the algorithm and also summarize various bounds that we need. We begin with some general bounds on the number $|\Sigma(n, t)|$ of magic squares and the density factors p and ϕ in (1.3.2) and (2.4.1).

(3.1) Theorem. *We have*

(1)

$$\binom{N + n^2 - 1}{n^2 - 1} \geq |\Sigma(n, t)| \geq \binom{N + n - 1}{n - 1}^{-2} \binom{N + n^2 - 1}{n^2 - 1};$$

(2)

$$1 \leq p(X) \leq \frac{(t!)^n N^N}{t^N N!} \quad \text{for all } X \in \Delta;$$

(3)

$$0 < \phi(X) \leq \frac{(N + n^2 - 1)! N!}{(n^2 - 1)! (t!)^{2n} n^{2N}} \quad \text{for all } X \in \Delta.$$

We will use somewhat cruder estimates which are easier to work with.

(3.2) Corollary. *We have*

(1)

$$\binom{N+n^2-1}{n^2-1} \geq |\Sigma(n,t)| \geq (N+n)^{-2n} \binom{N+n^2-1}{n^2-1};$$

(2)

$$1 \leq p(X) \leq (8t)^{n/2} \quad \text{for all } X \in \Delta;$$

(3)

$$0 \leq f(X) \leq \binom{N+n^2-1}{n^2-1} \quad \text{for all } X \in \Delta.$$

We prove Theorem 3.1 in Section 5. Corollary 3.2 follows by standard estimates via Stirling's formula

$$(3.2.1) \quad \sqrt{2\pi x} \left(\frac{x}{e}\right)^x e^{1/(12x+1)} \leq \Gamma(x+1) \leq \sqrt{2\pi x} \left(\frac{x}{e}\right)^x e^{1/(12x)} \quad \text{for } x \geq 0.$$

It turns out that most of the time we have $p(X) = N^{O(\ln N)}$. Now we state the key estimate of the paper.

(3.3) Theorem. *For any $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that for all positive integers n and t such that*

$$t < e^n,$$

we have

$$\mu \left\{ X \in \Delta_{n \times n} : p(X) \geq N^{\beta \ln N} \right\} < N^{-\alpha n}.$$

We prove Theorem 3.3 in Section 8 having established an estimate for the entries of the doubly stochastic scaling of a matrix in Section 6 and some standard probability bounds in Section 7.

Finally, we need some technical estimates showing that ϕ is sufficiently regular so that we can indeed apply integration and sampling algorithms of [AK91], [F+94], [FK99], and [LV06].

For

$$0 < \delta < \frac{1}{n^2},$$

let us define the δ -interior of the simplex by

$$\Delta^\delta = \Delta_{n \times n}^\delta = \left\{ X = (x_{ij}) : x_{ij} > \delta \quad \text{for all } i, j \quad \text{and} \quad \sum_{i,j=1}^n x_{ij} = 1 \right\}.$$

(3.4) Theorem. *We have*

(1)

$$\int_{\Delta^\delta} f d\mu \leq \int_{\Delta} f d\mu \leq (1 - \delta n^2)^{-N - n^2 + 1} \int_{\Delta^\delta} f d\mu,$$

(2) *For any $X, Y \in \Delta^\delta$ where $X = (x_{ij})$ and $Y = (y_{ij})$ we have*

$$|\ln \phi(X) - \ln \phi(Y)| \leq \frac{N}{\delta} \max_{ij} |x_{ij} - y_{ij}|.$$

Theorem 3.4 is proven in Section 4 of [Ba05]. For completeness, we present its straightforward proof in Section 9.

(3.5) The algorithm. Now we can describe the algorithm in more detail. First, we assume that $t < e^n$. Indeed, for $t \geq n^3$ there is a randomized polynomial time approximation scheme for computing $|\Sigma(n, t)|$ which is a particular case of the algorithm of Dyer, Kannan, and Mount [D+97], see also [Mo02] for a strengthening. Second, we assume that $\epsilon > N^{-n}$, which is not really restrictive since $|\Sigma(n, t)|$ can be computed exactly in $N^{O(n)}$ time by a dynamic programming algorithm.

For an $0 < \epsilon < 1$, let us choose

$$\delta = \frac{-\ln(1 - \epsilon)}{n^2(N + n^2 - 1)} \approx \frac{\epsilon}{n^2(N + n^2 - 1)} \quad \text{for small } \epsilon > 0.$$

By Part 1 of Theorem 3.4, the integral

$$(3.5.1) \quad \int_{\Delta^\delta} f d\mu$$

approximates $|\Sigma(n, t)|$ from below within the relative error ϵ . We factor $f = p\phi$ as in Section 2.4. By Parts 1 and 2 of Corollary 3.2 and Theorem 3.3 (with a sufficiently large α) it follows that for \bar{p} defined by

$$\bar{p}(X) = \begin{cases} p(X) & \text{if } p(X) \leq T \\ T & \text{if } p(X) > T \end{cases}$$

with $T = N^{\beta \ln N}$, the integral

$$(3.5.2) \quad \int_{\Delta^\delta} \bar{p}\phi d\mu$$

approximates (3.5.1) from below within relative error $N^{-n} < \epsilon$.

A simplified, polynomial time algorithm, replaces integral (3.5.2) by the integral

$$(3.5.3) \quad \int_{\Delta^\delta} \phi d\mu,$$

which approximates (3.5.2) within a factor of $N^{\beta \ln N}$. Because ϕ is log-concave and satisfies the bound of Part 2 of Theorem 3.4, the integral (3.5.3) can be computed within relative error ϵ by any of the algorithms of [AK91], [F+94], [FK99], and [LV06] in time polynomial in N and ϵ^{-1} . We use an algorithm of [L+00] to compute $\phi(x)$ for a given $x \in \Delta$.

In the more accurate, but also more time consuming, version of the algorithm, we write (3.5.2) as

$$\left(\int_{\Delta^\delta} \bar{p} \, d\nu \right) \left(\int_{\Delta^\delta} \phi \, d\mu \right),$$

where ν is the probability measure on Δ^δ with the density proportional to ϕ . Furthermore, the algorithms of [AK91], [F+94], [FK99], and [LV06] allow us to sample independent random points from a probability measure $\tilde{\nu}$ sufficiently close to $\tilde{\nu}$, that is satisfying

$$|\tilde{\nu}(S) - \nu(S)| < \epsilon N^{-\beta \ln N} \quad \text{for any Borel } S \subset \Delta^\delta.$$

A single point can be sampled in $(1/\epsilon)^{O(1)} N^{O(\ln N)}$ time. We sample $m = \lceil 3T^2 \epsilon^{-2} \rceil$ independent random points x_i with respect to measure $\tilde{\nu}$ and estimate the integral

$$(3.5.4) \quad \int_{\Delta^\delta} \bar{p} \, d\tilde{\nu}$$

by the sample mean

$$(3.5.5) \quad m^{-1} \sum_{i=1}^m \bar{p}(x_i)$$

By Chebyshev inequality, (3.5.5) approximates (3.5.4) within relative error ϵ with probability at least $2/3$. We use the algorithm [J+04] to compute $p(x_i)$.

(3.6) Details of the implementation. We implemented a much simplified version of the algorithm, computing the integral (1.3.4)

$$\int_{\Delta} \phi \, d\mu.$$

In our implementation, we work with the original simplex Δ , not its δ -interior Δ^δ . This has never given us any boundary-related trouble in our computational experiments.

The implementation is based on a version of the hit-and-run algorithm of [LV06], see also [Ve05]. We use telescoping with respect to the density ϕ . Namely, we pick a sufficiently dense uniform subset

$$0 < t_1 < t_2 < \dots < t_m = t,$$

and define a log-concave function ψ_i by

$$\psi_i(X) = \sigma^{t_i}(X) \quad \text{for } X \in \Delta,$$

cf. Section 2.4 and formula (2.4.1), in particular. The number m of points is chosen by the user so as to be “reasonable”.

For a given $X \in \Delta$ we compute $\sigma(X)$ by the Sinkhorn balancing (alternate scaling of rows and columns of X to the unit sums) [Si64], which seems to work very well even for n as large as $n = 100$.

Note that the numbers t_i are not necessarily integer and that

$$\phi(X) = \frac{(N + n^2 - 1)! N! t^N}{(n^2 - 1)! (t!)^{2n} N^N} \psi_m(X),$$

cf. (2.4.1). Hence the goal is to compute

$$\int_{\Delta} \psi_m d\mu = S_1 \prod_{i=1}^{m-1} \frac{S_{i+1}}{S_i} \quad \text{where}$$

$$S_i = \int_{\Delta} \psi_i d\mu \quad \text{for } i = 1, \dots, m.$$

If t_1 is sufficiently small, the function ψ_1 is close to a constant, so we estimate S_1 by a sample mean of ψ_1 for a set of randomly chosen $X \in \Delta$. In our experiments, we often chose $t_1 = 1$. To choose a random $X \in \Delta$ from μ , we choose the entries x_{ij} of X independently from the standard exponential distribution and then normalize:

$$x_{ij} := x_{ij} \left(\sum_{k,l=1}^n x_{kl} \right)^{-1} \quad \text{for } i, j = 1, \dots, n.$$

To compute ratios

$$\frac{S_{i+1}}{S_i} = \int_{\Delta} \frac{\psi_{i+1}}{\psi_i} d\nu_i,$$

where ν_i is the probability measure on Δ with the density proportional to ψ_i , we sample points $X \in \Delta$ from ν_i and average the ratios $\psi_{i+1}(X)/\psi_i(X)$. If t_{i+1} is sufficiently close to t_i , the ratios are close to 1, so a moderate number of sampled points is needed. Again, the number of samples is selected by the user. Thus the bottleneck of this simplified algorithm consists in sampling random points $X \in \Delta$ from the probability measure ν_i . For that, we iterate the basic “hit-and-run” construction. We sample a random point $X_0 \in \Delta$ from the uniform distribution μ , pick a random line ℓ through X in the affine hull of Δ , sample X_1 from the density on the interval $\ell \cap \Delta$ proportional to the restriction of ψ_i onto that interval and iterate the process with $X_0 := X_1$ as the new starting point. After a number of

iterations chosen by the user, the point X_1 is accepted as a random sample from the measure ν_i .

To choose a random line ℓ through X we first sample an $n \times n$ matrix $L = (\lambda_{ij})$ of independent standard Gaussian random variables and then center it

$$\lambda_{ij} := \lambda_{ij} - \frac{1}{n^2} \sum_{k,l=1}^n \lambda_{kl} \quad \text{for } i, j = 1, \dots, n.$$

We then define the line ℓ through X by

$$\ell = \left\{ X + \tau L : \tau \in \mathbb{R} \right\}.$$

To choose a point $X_1 \in \ell \cap \Delta$, we approximate the restriction of ψ_i onto $\ell \cap \Delta$ by a function $\widetilde{\psi}_{i,\ell}$ such that $\ln \widetilde{\psi}_{i,\ell}$ is a piece-wise linear approximation of the restriction of $\ln \psi_i$ onto $\ell \cap \Delta$. Sampling from the density proportional to $\widetilde{\psi}_{i,\ell}$ reduces then to sampling from the exponential distribution.

4. PRELIMINARIES: ESTIMATES ON THE PERMANENT

We will use the following bounds for the permanent.

(4.1) The van der Waerden bound. Let $B = (b_{ij})$ be an $N \times N$ doubly stochastic matrix, that is,

$$\sum_{j=1}^N b_{ij} = 1 \quad \text{for } i = 1, \dots, N \quad \text{and} \quad \sum_{i=1}^N b_{ij} = 1 \quad \text{for } j = 1, \dots, N$$

and

$$b_{ij} \geq 0 \quad \text{for } i, j = 1, \dots, N.$$

Then

$$\text{per } B \geq \frac{N!}{N^N}.$$

This is the famous van der Waerden bound proved by Falikman [Fa81] and Egorichev [Eg81], see also Chapter 12 of [LW01].

(4.2) The continuous version of the Bregman-Minc bound. Let $B = (b_{ij})$ be an $N \times N$ matrix such that

$$\sum_{j=1}^N b_{ij} \leq 1 \quad \text{for } i = 1, \dots, N$$

and

$$b_{ij} \geq 0 \quad i, j = 1, \dots, N.$$

Furthermore, let

$$s_i = \max_{j=1, \dots, N} b_{ij} > 0 \quad \text{for } i = 1, \dots, N.$$

Then

$$\text{per } B \leq \prod_{i=1}^N s_i \Gamma^{s_i} \left(\frac{1 + s_i}{s_i} \right).$$

This bound was obtained by Soules [So03].

If $s_i = 1/r_i$ for integers r_i the bound transforms into

$$\text{per } B \leq \prod_{i=1}^N \frac{(r_i!)^{1/r_i}}{r_i},$$

which can be easily deduced from the Minc conjecture proved by Bregman [Br73], see also Chapter 11 of [LW01].

We will use the following corollary of estimates of Sections 4.1 – 4.2.

(4.3) Corollary. *Let $B = (b_{ij})$ be an $N \times N$ doubly stochastic matrix and let*

$$s_i = \max_{j=1, \dots, N} b_{ij} \quad \text{for } i = 1, \dots, N.$$

Suppose that

$$\sum_{i=1}^N s_i \leq \gamma \quad \text{for some } \gamma \geq 1.$$

Then

$$\begin{aligned} \frac{N!}{N^N} \leq \text{per } B &\leq \left(\frac{\gamma}{N} \right)^N \Gamma^\gamma \left(1 + \frac{N}{\gamma} \right) \\ &\leq \frac{N!}{N^N} (2\pi N)^{\gamma/2} e^{\gamma^2/12N}. \end{aligned}$$

Proof. The lower bound is the van der Waerden estimate, see Section 4.1.

Let us define

$$g(\xi) = \xi \ln \Gamma \left(\frac{1 + \xi}{\xi} \right) + \ln \xi \quad \text{for } 0 < \xi \leq 1.$$

Then g is a concave function, cf. [So03], and by the inequality of Section 4.2, we have

$$\ln \text{per } B \leq \sum_{i=1}^N g(s_i).$$

The function

$$G(x) = \sum_{i=1}^N g(\xi_i) \quad \text{for } x = (\xi_1, \dots, \xi_N)$$

is concave on the simplex defined by the equation $\xi_1 + \dots + \xi_N = \gamma$ and inequalities $\xi_i \geq 0$ for $i = 1, \dots, N$. It is also symmetric under permutations of ξ_1, \dots, ξ_N . Hence the maximum of G is attained at

$$\xi_1 = \dots = \xi_N = \gamma/N,$$

and so

$$\ln \text{per } B \leq Ng \left(\frac{\gamma}{N} \right).$$

Thus

$$\text{per } B \leq \left(\frac{\gamma}{N} \right)^N \Gamma^\gamma \left(1 + \frac{N}{\gamma} \right)$$

and the rest follows by Stirling's formula (3.2.1). \square

We will apply Corollary 4.3 for $\gamma = O(\ln N)$, in which case the ratio of the upper and lower bounds is $N^{O(\ln N)}$.

5. PROOF OF THEOREM 3.1

To prove the upper bound in Part 1, we note that $|\Sigma(n, t)|$ does not exceed the number of $n \times n$ non-negative integer matrices with the sum of entries equal to N , which is exactly equal to $\binom{N+n^2-1}{n^2-1}$.

For non-negative integer vectors $R = (r_1, \dots, r_n)$ and $C = (c_1, \dots, c_n)$ such that $r_1 + \dots + r_n = c_1 + \dots + c_n = N$, let $|\Sigma(R, C)|$ be the number of $n \times n$ non-negative integer matrices with the row sums r_1, \dots, r_n and the column sums c_1, \dots, c_n , cf. Section 1.2. As is discussed in [Ba07], see formula (4) of Section 2 there, we have

$$|\Sigma(R, C)| \leq |\Sigma(n, t)| \quad \text{for all } R \text{ and } C.$$

Since the total number of pairs (R, C) is $\binom{N+n-1}{n-1}^2$, the lower bound follows by summing up the previous inequality over all choices of (R, C) .

We recall that p is defined by

$$p(X) = \frac{N^N}{N!} \text{per}(Y \otimes J_t),$$

where Y is an $n \times n$ doubly stochastic scaling of X and J_t is the $t \times t$ matrix filled with $1/t$, cf. Section 2.4. Hence $Y \otimes J_t$ is an $N \times N$ doubly stochastic matrix and the lower bound in Part 2 follows by the van der Waerden bound, see Section 4.1. Since the entries of $Y \otimes J_t$ do not exceed $1/t$, by Section 4.2, we have

$$p(x) \leq \frac{N^N}{N!} \frac{(t!)^n}{t^N},$$

which completes the proof of Part 2.

Clearly,

$$\phi(X) = \frac{(N + n^2 - 1)!N!t^N}{(n^2 - 1)!(t!)^{2n}N^N} \sigma^t(X) > 0 \quad \text{for all } X \in \Delta.$$

Since $\sigma(X)$ is log-concave and invariant under permutations of rows and columns of X , the maximum of σ on Δ is attained at the $n \times n$ matrix A with the entries $1/n^2$ that is the average of all the matrices obtained from any given $X \in \Delta$ by permutations of rows and columns. It is immediate that $\sigma(A) = n^{-n}$ and hence

$$\phi(X) \leq \phi(A) = \frac{(N + n^2 - 1)!N!t^N}{(n^2 - 1)!(t!)^{2n}N^N} n^{-N},$$

and the proof of Part 3 follows. \square

As we remarked before, Corollary 3.2 is obtained by a straightforward application of the Stirling formula and a remark that $f = p\phi$.

6. BOUNDING THE ENTRIES OF THE DOUBLY STOCHASTIC SCALING OF A MATRIX

In this section we prove the following main result.

(6.1) Theorem. *Let $A = (a_{ij})$ be an $n \times n$ positive matrix and let $B = (b_{ij})$ be the doubly stochastic scaling of A so that for some $\lambda_i, \mu_j > 0$ we have*

$$a_{ij} = b_{ij}\lambda_i\mu_j \quad \text{for all } i, j \quad \text{and}$$

$$\sum_{j=1}^n b_{ij} = 1 \quad \text{for } i = 1, \dots, n \quad \text{and} \quad \sum_{i=1}^n b_{ij} = 1 \quad \text{for } j = 1, \dots, n.$$

Then, for all k and l ,

$$\begin{aligned} \ln b_{kl} &\leq \ln a_{kl} - \frac{1}{n-2} \sum_{j \neq l} \ln a_{kj} - \frac{1}{n-2} \sum_{i \neq k} \ln a_{il} \\ &\quad + \frac{n}{n-2} \ln \left(\frac{1}{n} \sum_{i,j=1}^n a_{ij} \right) - \frac{2n-2}{n-2} \ln(n-1). \end{aligned}$$

Example. Suppose that $1 \leq a_{ij} \leq 2$ for all i, j . Theorem 6.1 implies that for some absolute constant γ we have

$$b_{ij} \leq \frac{\gamma}{n} \quad \text{for all } i, j.$$

Hence by Corollary 4.3,

$$\frac{n!}{n^n} \leq \text{per } B \leq n^{O(1)} \frac{n!}{n^n}$$

and hence values of $\text{per } B$ vary within up to a polynomial in n factor. In contrast,

$$n! \leq \text{per } A \leq 2^n n!,$$

so values of $\text{per } A$ vary within an exponential in n factor.

This concentration of the permanent of the doubly stochastic scaling of a matrix is the basis of our approach.

The proof of Theorem 6.1 is based on the following two lemmas.

The first lemma was proved in [L+00], for completeness we include its proof here.

(6.2) Lemma. *Let $A = (a_{ij})$ be an $n \times n$ positive matrix such that*

$$\sum_{i,j=1}^n a_{ij} = n$$

and let $B = (b_{ij})$ be the doubly stochastic scaling of A . Then

$$\sum_{i,j=1}^n \ln b_{ij} \geq \sum_{i,j=1}^n \ln a_{ij}.$$

Proof. As is known, B can be obtained from A as the limit of repeated alternate scalings of the rows of A to the row sums equal to 1 and of the columns of A to the column sums equal to 1 (Sinkhorn balancing), see [Si64]. Hence it suffices to prove that under the row (column) scalings, the sum of the logarithms of the entries of the matrix can only increase.

To this end, let $C = (c_{ij})$ be a positive $n \times n$ matrix with the row sums ρ_1, \dots, ρ_n such that $\rho_1 + \dots + \rho_n = n$ and let $D = (d_{ij})$ be the matrix such that

$$d_{ij} = \frac{c_{ij}}{\rho_i} \quad \text{for all } i, j.$$

In words: we divide the i th row of C by its row sum ρ_i . We note that the sum of the entries of D is n and that

$$\sum_{i,j=1}^n (\ln d_{ij} - \ln c_{ij}) = -n \sum_{i=1}^n \ln \rho_i \geq 0$$

because of the arithmetic-geometric mean inequality.

Column scalings are handled in the same way. \square

The following result is obtained in [Br73]. For completeness, we provide its proof below.

(6.3) Lemma. Let $A = (a_{ij})$ be a positive $n \times n$ matrix and let $B = (b_{ij})$ be its doubly stochastic scaling. Then B is the solution of the optimization problem

$$\text{minimize } \sum_{i,j=1}^n x_{ij} (\ln x_{ij} - \ln a_{ij})$$

over the set of all $n \times n$ doubly stochastic matrices $X = (x_{ij})$.

Proof. First, we note the minimum is attained on a positive doubly stochastic matrix X . If, for example, $x_{11} = 0$, then there are indices i and j such that $x_{i1} > 0$, $x_{1j} > 0$, $x_{ij} < 1$ and one can make the value of the objective function smaller by modifying

$$x_{11} := \epsilon, \quad x_{1j} := x_{1j} - \epsilon, \quad x_{i1} := x_{i1} - \epsilon, \quad x_{ij} := x_{ij} + \epsilon$$

for a sufficiently small $\epsilon > 0$. This follows since the right derivative of $x \ln x$ is $-\infty$ at $x = 0$ and is finite at any $x > 0$.

Since the optimal point X lies in the relative interior of the set of doubly stochastic matrices, the gradient of the objective function at X should be orthogonal to the space of $n \times n$ matrices with the row and column sums equal to 0.

This gives us the following equations

$$\ln x_{ij} - \ln a_{ij} = \xi_i + \eta_j \quad \text{for all } i, j$$

and some numbers $\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n$.

In other words,

$$x_{ij} = a_{ij} \lambda_i \mu_j \quad \text{for } \lambda_i = e^{\xi_i} \quad \text{and} \quad \mu_j = e^{\eta_j} \quad \text{for all } i, j$$

as desired. □

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. First, we notice that neither the matrix B nor the right hand side of the inequality change if we scale

$$a_{ij} := a_{ij} \tau \quad \text{for all } i, j$$

and some $\tau > 0$. Therefore, without loss of generality, we assume that

$$\sum_{i,j=1}^n a_{ij} = n.$$

Without loss of generality, we assume that $k = l = 1$, so our goal is to bound b_{11} .

By Lemma 6.3, matrix B is the solution of the minimization problem

$$\text{minimize } \sum_{i,j=1}^n x_{ij} (\ln x_{ij} - \ln a_{ij})$$

over the set of $n \times n$ doubly stochastic matrices $X = (x_{ij})$.

For a real τ , let us define the matrix $B(\tau) = (b_{ij}(\tau))$ by

$$b_{ij}(\tau) = \begin{cases} b_{11} + \tau & \text{if } i = j = 1, \\ b_{1j} - \tau/(n-1) & \text{if } i = 1, j \neq 1, \\ b_{i1} - \tau/(n-1) & \text{if } i \neq 1, j = 1, \\ b_{ij} + \tau/(n-1)^2 & \text{if } i \neq 1, j \neq 1. \end{cases}$$

We observe that $B(0) = B$, that the row and column sums of $B(\tau)$ are 1 and that $B(\tau)$ is positive for all τ from a sufficiently small neighborhood of the origin. Therefore, if we let

$$f(\tau) = \sum_{i,j=1}^n b_{ij}(\tau) (\ln b_{ij}(\tau) - \ln a_{ij}),$$

we must have

$$f'(0) = 0.$$

Computing the derivative, we get

$$\begin{aligned} f'(0) &= \ln b_{11} - \ln a_{11} + 1 \\ &\quad - \frac{1}{n-1} \sum_{j \neq 1} (\ln b_{1j} - \ln a_{1j}) - 1 \\ &\quad - \frac{1}{n-1} \sum_{i \neq 1} (\ln b_{i1} - \ln a_{i1}) - 1 \\ &\quad + \frac{1}{(n-1)^2} \sum_{i,j \neq 1} (\ln b_{ij} - \ln a_{ij}) + 1. \end{aligned}$$

Rearranging summands, we rewrite the derivative in the form

$$\begin{aligned} f'(0) &= \left(1 - \frac{1}{(n-1)^2}\right) (\ln b_{11} - \ln a_{11}) \\ &\quad - \left(\frac{1}{n-1} + \frac{1}{(n-1)^2}\right) \sum_{j \neq 1} (\ln b_{1j} - \ln a_{1j}) \\ &\quad - \left(\frac{1}{n-1} + \frac{1}{(n-1)^2}\right) \sum_{i \neq 1} (\ln b_{i1} - \ln a_{i1}) \\ &\quad + \frac{1}{(n-1)^2} \sum_{i,j=1}^n (\ln b_{ij} - \ln a_{ij}). \end{aligned}$$

Since $f'(0) = 0$ and since by Lemma 6.2 we have

$$\sum_{i,j=1}^n \ln b_{ij} \geq \sum_{i,j=1}^n \ln a_{ij},$$

we must have

$$\begin{aligned} & \frac{n^2 - 2n}{(n-1)^2} (\ln b_{11} - \ln a_{11}) \\ & - \frac{n}{(n-1)^2} \sum_{j \neq 1} (\ln b_{1j} - \ln a_{1j}) - \frac{n}{(n-1)^2} \sum_{i \neq 1} (\ln b_{i1} - \ln a_{i1}) \\ & \leq 0. \end{aligned}$$

That is,

$$(n-2)(\ln b_{11} - \ln a_{11}) - \sum_{j \neq 1} (\ln b_{1j} - \ln a_{1j}) - \sum_{i \neq 1} (\ln b_{i1} - \ln a_{i1}) \leq 0.$$

In other words,

$$\begin{aligned} & \ln b_{11} - \frac{1}{n-2} \sum_{j \neq 1} \ln b_{1j} - \frac{1}{n-2} \sum_{i \neq 1} \ln b_{i1} \\ & \leq \ln a_{11} - \frac{1}{n-2} \sum_{j \neq 1} \ln a_{1j} - \frac{1}{n-2} \sum_{i \neq 1} \ln a_{i1} \end{aligned}$$

and

$$\begin{aligned} \ln b_{11} & \leq \ln a_{11} - \frac{1}{n-2} \sum_{j \neq 1} \ln a_{1j} - \frac{1}{n-2} \sum_{i \neq 1} \ln a_{i1} \\ & \quad + \frac{1}{n-2} \sum_{j \neq 1} \ln b_{1j} + \frac{1}{n-2} \sum_{i \neq 1} \ln b_{i1}. \end{aligned}$$

On the other hand, if the value of b_{11} is fixed, the maximum value of

$$\sum_{j \neq 1} \ln b_{1j} + \sum_{i \neq 1} \ln b_{i1}$$

is attained at

$$b_{1j} = b_{i1} = \frac{1 - b_{11}}{n-1} \quad \text{for all } i, j \neq 1$$

(since the row and column sums of B are equal to 1).

Therefore, we have

$$\begin{aligned} \ln b_{11} & \leq \ln a_{11} - \frac{1}{n-2} \sum_{j \neq 1} \ln a_{1j} - \frac{1}{n-2} \sum_{i \neq 1} \ln a_{i1} \\ & \quad + \frac{2n-2}{n-2} \ln(1 - b_{11}) - \frac{2n-2}{n-2} \ln(n-1). \end{aligned}$$

Since $\ln(1 - b_{11}) \leq 0$ and $\sum_{i,j=1}^n a_{ij} = n$, this completes the proof. \square

7. PROBABILISTIC ESTIMATES

The goal of this section is to prove the following technical estimates.

(7.1) Lemma.

- (1) Let a_1, \dots, a_n be independent standard exponential random variables. Then for every $r > 0$ we have

$$\mathbf{P} \left\{ \frac{1}{n} \sum_{i=1}^n \ln a_i \leq -r \right\} \leq \left(\frac{\pi}{e^r} \right)^{n/2};$$

- (2) Let a_1, \dots, a_m be independent standard exponential random variables. Then for every $r > 0$ we have

$$\mathbf{P} \left\{ \frac{1}{m} \sum_{i=1}^m a_i \geq r \right\} \leq \left(\frac{4}{e^r} \right)^{m/2}.$$

- (3) Let a_{ij} , $1 \leq i, j \leq n$, be independent standard exponential random variables and let

$$c_i = \max_{j=1, \dots, n} a_{ij} \quad \text{for } i = 1, \dots, n.$$

Then for every $r > 0$ we have

$$\mathbf{P} \left\{ \frac{1}{n} \sum_{i=1}^n c_i > r \right\} \leq e^{-rn/2} (n\sqrt{\pi})^n.$$

Proof. We use the Laplace transform method, see, for example, Appendix A of [AS00].

To prove Part 1, let

$$b = \frac{1}{n} \sum_{i=1}^n \ln a_i.$$

For $\tau > 0$ we get

$$\mathbf{P} \{ b < -r \} = \mathbf{P} \{ e^{-\tau b} > e^{\tau r} \} \leq e^{-\tau r} \mathbf{E} e^{-\tau b}$$

by the Markov inequality. Let us choose $\tau = n/2$. Then

$$\mathbf{E} e^{-\tau b} = \mathbf{E} \prod_{i=1}^n a_i^{-1/2} = \left(\int_0^{+\infty} x^{-1/2} e^{-x} dx \right)^n = \Gamma^n(1/2) = \pi^{n/2}.$$

Hence

$$\mathbf{P} \left\{ \frac{1}{n} \sum_{i=1}^n \ln a_i < -r \right\} \leq \pi^{n/2} e^{-rn/2}.$$

To prove Part 2, let

$$b = \frac{1}{m} \sum_{i=1}^m a_i.$$

For $\tau > 0$ we get

$$\mathbf{P}\{b > r\} = \mathbf{P}\{e^{\tau b} > e^{\tau r}\} \leq e^{-\tau r} \mathbf{E} e^{\tau b}.$$

Let us choose $\tau = m/2$. Then

$$\mathbf{E} e^{\tau b} = \prod_{i=1}^m \mathbf{E} e^{a_i/2} = \left(\int_0^{+\infty} e^{-x/2} dx \right)^m = 2^m.$$

To prove Part 3, let

$$b = \frac{1}{n} \sum_{i=1}^n c_i.$$

For $\tau > 0$ we get

$$\mathbf{P}\{b > r\} \leq e^{-\tau r} \mathbf{E} e^{\tau b}.$$

We choose $\tau = m/2$. Then

$$\mathbf{E} e^{\tau b} = \prod_{i=1}^n \mathbf{E} e^{c_i/2}.$$

We have

$$\begin{aligned} \mathbf{E} \left(e^{c_i/2} \right) &= \mathbf{E} \left(\max_{j=1, \dots, n} e^{a_{ij}/2} \right) \leq \mathbf{E} \left(\sum_{j=1}^n e^{a_{ij}/2} \right) \\ &= n \int_0^{+\infty} e^{-x/2} dx = n\Gamma(1/2) = n\sqrt{\pi} \end{aligned}$$

and the proof follows. \square

8. PROOF OF THEOREM 3.3

Let $\text{Mat}_+ = \text{Mat}_+(n, n)$ be the set of $n \times n$ positive matrices $A = (a_{ij})$ and let us consider the projection $\Psi : \text{Mat}_+(n, n) \rightarrow \Delta_{n \times n}$, where $\Psi(A) = X = (x_{ij})$ is defined by

$$x_{ij} = a_{ij} \left(\sum_{k,l=1}^n a_{kl} \right)^{-1} \quad \text{for } i, j = 1, \dots, n.$$

As is known and easy to check, the push-forward of the exponential measure ν on Mat_+ with the density

$$\exp \left\{ - \sum_{i,j=1}^n a_{ij} \right\}$$

is the probability measure μ on Δ . In other words, if A is a random $n \times n$ matrix with independent standard exponential entries then $X = \Psi(A)$ is a random matrix from the simplex Δ sampled in accordance with the uniform probability measure μ . Furthermore, the doubly stochastic scalings of A and $\Psi(A)$ coincide.

Let us choose $A \in \text{Mat}_+$, let $X = \Psi(A)$, and let B be the doubly stochastic scaling of A . Then

$$p(X) = \frac{N^N}{N!} \text{per}(B \otimes J_t).$$

In view of Corollary 4.3, the proof of Theorem 3.3 follows from the following result.

(8.1) Proposition. *For any $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that for all positive integers n and t such that*

$$t < e^n$$

the following holds.

Let $A = (a_{ij})$ be the $n \times n$ random matrix with the independent standard exponential entries and let $B = B(A)$, $B = (b_{ij})$, be its doubly stochastic scaling. Then

$$\mathbf{P} \left\{ \sum_{i=1}^n \left(\max_{j=1, \dots, n} b_{ij} \right) > \beta \ln N \right\} < N^{-\alpha n}.$$

Proof of Proposition 8.1. Let us introduce random variables

$$u_i = \frac{1}{n} \sum_{j=1}^n \ln a_{ij} \quad \text{for } i = 1, \dots, n \quad \text{and}$$

$$v_j = \frac{1}{n} \sum_{i=1}^n \ln a_{ij} \quad \text{for } j = 1, \dots, n.$$

Applying Part 1 of Lemma 7.1, we conclude that for some absolute constant $r > 0$ we have

$$\mathbf{P}\{u_i \leq -r\} \leq 2^{-n} \quad \text{and} \quad \mathbf{P}\{v_j \leq -r\} \leq 2^{-n} \quad \text{for all } i, j = 1, \dots, n.$$

It follows then that one can choose a $\beta_1 = \beta_1(\alpha) > 0$ such that

$$(8.1.1) \quad \mathbf{P}\left\{ |i : u_i < -r| > \beta_1 \ln N \right\} < \frac{1}{6} N^{-\alpha n} \quad \text{and}$$

$$\mathbf{P}\left\{ |j : v_j < -r| > \beta_1 \ln N \right\} < \frac{1}{6} N^{-\alpha n}.$$

Applying Part 2 of Lemma 7.1 with $m = n^2$ and using that $t < e^n$ we conclude that for some constant $\beta_2 = \beta_2(\alpha) > 0$ we have

$$(8.1.2) \quad \mathbf{P} \left\{ \frac{1}{n^2} \sum_{i,j=1}^n a_{ij} > \beta_2 \right\} < \frac{1}{3} N^{-\alpha n}.$$

Let us define

$$c_i = \max_{j=1, \dots, n} a_{ij}.$$

By Part 3 of Lemma 7.1, for some absolute constant $\beta_3 = \beta_3(\alpha) > 0$ we have

$$(8.1.3) \quad \mathbf{P} \left\{ \frac{1}{n} \sum_{i=1}^n c_i > \beta_3 \ln N \right\} < \frac{1}{3} N^{-\alpha n}.$$

Let us define a set $\mathcal{A} \subset \text{Mat}_+$ of matrices by

$$\mathcal{A} = \left\{ A = (a_{ij}) : \begin{array}{l} |i : u_i < -r| \leq \beta_1 \ln N, \quad |j : v_j < -r| \leq \beta_1 \ln N, \\ \frac{1}{n^2} \sum_{i,j=1}^n a_{ij} \leq \beta_2, \quad \text{and} \\ \frac{1}{n} \sum_{i=1}^n c_i \leq \beta_3 \ln N \end{array} \right\}.$$

From (8.1.1)–(8.1.3) we conclude

$$\mathbf{P}\{A \in \mathcal{A}\} \geq 1 - N^{-\alpha n}.$$

Let us pick a matrix $A \in \mathcal{A}$ and let B be its doubly stochastic scaling. By Theorem 6.1, we have

$$(8.1.4) \quad \begin{aligned} \ln b_{ij} &\leq \frac{n}{n-2} \ln a_{ij} - \frac{1}{n-2} u_i - \frac{1}{n-2} v_j \\ &\quad + \frac{n}{n-2} \ln \left(\frac{1}{n} \sum_{i,j=1}^n a_{ij} \right) - \frac{2n-2}{n-2} \ln(n-1). \end{aligned}$$

Let us define

$$I = \{i : u_i < -r\} \quad \text{and} \quad J = \{j : v_j < -r\},$$

so $|I|, |J| \leq \beta_1 \ln N$. Thus from (8.1.2) and (8.1.4) we deduce that for some constant $\gamma = \gamma(\beta_1, \beta_2, \beta_3)$ we have

$$b_{ij} \leq \frac{\gamma}{n} a_{ij}^{n/(n-2)} \leq \frac{\gamma}{n} c_i^{n/(n-2)} \quad \text{for } i \notin I, j \notin J.$$

To complete the proof, we use the estimates

$$\begin{aligned} \max_{j=1, \dots, n} b_{ij} &\leq 1 \quad \text{for } i \in I \quad \text{and} \\ \max_{j=1, \dots, n} b_{ij} &\leq \frac{\gamma}{n} c_i^{n/(n-2)} + \sum_{j \in J} b_{ij} \quad \text{for } i \notin I. \end{aligned}$$

Summarizing,

$$\begin{aligned} \sum_{i=1}^n \left(\max_{j=1, \dots, n} b_{ij} \right) &\leq |I| + \frac{\gamma}{n} \sum_{i=1}^n c_i^{n/(n-2)} + \sum_{i \notin I} \sum_{j \in J} b_{ij} \\ &\leq |I| + \frac{\gamma}{n} \left(\sum_{i=1}^n c_i \right)^{n/(n-2)} + |J| \\ &\leq \beta \ln N \end{aligned}$$

for some $\beta = \beta(\beta_1, \beta_3, \gamma)$ as desired. \square

9. PROOF OF THEOREM 3.4

We use that both f and ϕ are *positive homogeneous* of degree N , that is,

$$\begin{aligned} f(\lambda X) &= \lambda^N f(x) \quad \text{and} \quad \phi(\lambda X) = \lambda^N \phi(X) \\ \text{for all } X &\in \text{Mat}_+(n, n) \quad \text{and} \quad \lambda > 0 \end{aligned}$$

and *monotone*, that is

$$\begin{aligned} f(X) &\leq f(Y) \quad \text{and} \quad \phi(X) \leq \phi(Y) \\ \text{for all } X, Y &\in \text{Mat}_+(n, n), \quad X = (x_{ij}), \quad Y = (y_{ij}) \\ \text{such that } x_{ij} &\leq y_{ij} \quad \text{for } i, j = 1, \dots, n. \end{aligned}$$

Among these properties only the monotonicity of ϕ is not immediately obvious. It follows, for example, from the following representation of $\sigma(X)$, see [MO86] and Section 2.4. For a positive matrix $X = (x_{ij})$ we have

$$\begin{aligned} n^n \sigma(X) &= \left(\min_{i,j=1}^n \sum_{i,j=1}^n x_{ij} \xi_i \eta_j \right)^n \quad \text{over all } \xi_i, \eta_j > 0 \\ \text{subject to } \prod_{i=1}^n \xi_i &= \prod_{j=1}^n \eta_j = 1. \end{aligned}$$

Let dx be the Lebesgue measure on the hyperplanes $\sum_{i,j=1}^n x_{ij} = \text{const}$ in the space of all $n \times n$ matrices $X = (x_{ij})$. Using that $f(X)$ is homogeneous, we get

$$\int_{(1-\delta n^2)\Delta} f(x) dx = (1 - \delta n^2)^{N+n^2-1} \int_{\Delta} f(x) dx.$$

On the other hand, for all $X \in (1 - \delta n^2)\Delta$ the matrix $Y = (y_{ij})$ defined by $y_{ij} = x_{ij} + \delta$ lies in Δ_δ and $f(Y) \geq f(X)$, which completes the proof of Part 1.

To prove Part 2, let

$$\alpha = \max_{i,j} |x_{ij} - y_{ij}|.$$

Hence

$$\begin{aligned} x_{ij} &\leq y_{ij} + \alpha \leq (1 + \alpha/\delta) y_{ij} \quad \text{and} \\ y_{ij} &\leq x_{ij} + \alpha \leq (1 + \alpha/\delta) x_{ij} \quad \text{for all } i, j \end{aligned}$$

Using monotonicity and homogeneity of ϕ we conclude that

$$\phi(X) \leq (1 + \alpha/\delta)^N \phi(Y) \quad \text{and} \quad \phi(Y) \leq (1 + \alpha/\delta)^N \phi(X),$$

from which the proof follows. \square

10. CONCLUDING REMARKS

(10.1) Counting general contingency tables. It is plausible to attempt to devise similar algorithms for counting contingency tables with the given row and column sums $R = (r_1, \dots, r_m)$ and $C = (c_1, \dots, c_n)$, where

$$r_1 + \dots + r_m = c_1 + \dots + c_n = N,$$

cf. Section 1. While the general idea can be easily generalized to this case, cf. [Ba05], we were unable so far to prove all the necessary bounds, except in the special case when the row sums are equal

$$r_1 = \dots = r_m = t$$

or the column sums are equal

$$c_1 = \dots = c_n = t,$$

but not necessarily both.

Suppose, for example, that the row sums are equal. Modifying the construction slightly, one can represent the required number of tables by the integral

$$\int_Q f \, d\mu,$$

where Q is the set of non-negative $m \times n$ matrices with all the row sums equal to 1 (geometrically, Q is a product of m simplices of dimension $(n-1)$ each) and μ is the Lebesgue measure on Q normalized by the condition $\mu(Q) = 1$. The function f factors into the product $f = p\phi$ of a log-concave function ϕ and a slowly varying function p and all the necessary estimate can be carried through, resulting in a randomized polynomial time algorithm approximating the number of tables within a factor of $N^{\log N}$ and a randomized quasi-polynomial algorithm of $(1/\epsilon)^{O(1)} N^{\log N}$ complexity to approximate the number of tables within any given relative error $\epsilon > 0$.

(10.2) Improving the bound. The bottleneck of our algorithm is defined by the ratio

$$c(n, t) = |\Sigma(n, t)| / \left(\int_{\Delta} \phi \, d\mu \right),$$

where ϕ is the log-concave density on the simplex Δ defined by (2.4.1). Roughly speaking, $c(n, t)$ is the main contribution to the computational complexity. We proved that $c(n, t) = N^{O(\ln N)}$ and some conjectural inequalities for the permanent (Conjectures 1.1 and 1.6 of [Sa06]) imply that we can choose the threshold $T = N^{O(1)}$ in Section 1.3 and therefore one should have $c(n, t) = N^{O(1)}$.

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