On linear programming bounds for spherical codes and designs

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Abstract

We investigate universal bounds on spherical codes and spherical designs that could be obtained using Delsarte's linear programming methods. We give a *lower* estimate for the LP upper bound on codes, and an *upper* estimate for the LP lower bound on designs.

Specifically, when the distance of the code is fixed and the dimension goes to infinity, the LP upper bound on codes is at least as large as the average of the best known upper and lower bounds.

When the dimension n of the design is fixed, and the strength k goes to infinity, the LP bound on designs turns out, in conjunction with known lower bounds, to be proportional to k^{n-1}.

1 Introduction

An n-dimensional spherical code of (angular) distance θ is a subset of the $(n-1)$-dimensional unit sphere, such that the angle between any two distinct points is at least θ. Equivalently, the Euclidean distance between any two distinct points is at least $2\sin(\theta/2)$.

An n-dimensional spherical design of strength k is a finite subset W of the $(n-1)$-dimensional unit sphere, such that for any algebraic polynomial f of n variables and degree k we have

$$\int_{S^{n-1}} f(x)dx = \frac{1}{|W|}\sum_{u\in W} f(u).$$

We are interested in the maximal cardinality $M(n,\theta)$ of a spherical code of distance θ, and in the minimal cardinality $N(n,k)$ of a design of strength k.

For a fixed θ and $n\to\infty$, $M(n,\theta)$ increases exponentially in n. The best known existential (lower) bound on the exponent $\frac{1}{n}\log M(n,\theta)$ is obtained by a volume argument [4]:

$$\frac{1}{n}\log M(n,\theta) \geq \log\frac{1}{\sin\theta} - o(1),$$

as n goes to infinity.

For a fixed n and $k\to\infty$, $N(n,k)$ increases polynomially in k. The best known existential (upper) bound on $N(n,k)$ is [7]

$$N(n,k) \leq O\left(k^{\frac{n(n-1)}{2}}\right),$$

as k goes to infinity.

The best universal bounds on codes and designs (upper for codes and lower for designs) are obtained using linear programming methods, initiated by Delsarte [5].

Let $\{P_s^{\alpha,\beta}\}$ be the Jacobi polynomials, orthogonal with respect to a weight function $w^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ on $(-1,1)$. For $\alpha = \beta = \frac{n-3}{2}$, we will simply write $\{P_s\},\ w(t)$. We will assume the standard normalization [12], in particular $P_0 \equiv 1$. Then [8], [6]

$$M(n,\theta) \le \min\left\{F(1):\ F = \sum_{s=1}^{m} a_s P_s;\ a_s \ge 0,\ a_0 = 1;\ F \le 0 \text{ on } [-1,\cos\theta]\right\} \tag{1}$$

And [6]

$$N(n,k) \ge \max\left\{\frac{1}{a_0}:\ F = \sum_{s=1}^{m} a_s P_s;\ F \ge 0 \text{ on } [-1,1],\ F(1) = 1;\ a_s \le 0 \text{ for } s \ge k\right\}. \tag{2}$$

In (1) and (2) the degree m of the polynomial F may be arbitrarily large.

We will denote the RHS of (1) by $M_{LP}(n,\theta)$ and the RHS of (2) by $N_{LP}(n,t)$.

Kabatyansky and Levenshtein [8] obtain the best known upper bound on $M(n,\theta)$

$$\frac{1}{n}\log M_{LP}(n,\theta) \le \frac{1+\sin\theta}{2\sin\theta}\log\frac{1+\sin\theta}{2\sin\theta} - \frac{1-\sin\theta}{2\sin\theta}\log\frac{1-\sin\theta}{2\sin\theta} + o(1).$$

Yudin [13] gives the best known lower bound on $N(n,k)$, for n fixed and $k \to \infty$

$$N_{LP}(n,k) \ge \frac{\int_{-1}^{1} w(t)dt}{\int_{\gamma}^{1} w(t)dt},$$

where γ is the maximal root of $P_{k-1}^{\frac{n-1}{2},\frac{n-1}{2}}$. For n fixed and $k \to \infty$, this is at least ([9], pp. 117-120)

$$\Omega\left(2^{-c(4n)^{\frac{1}{3}}} \cdot \left(\frac{2}{n}\right)^n \cdot k^{n-1}\right),$$

where $c \approx 1.86$, improving the lower bound of Delsarte, Goethals, and Seidel [6] by a factor of $\left(\frac{4}{e}\right)^n \cdot 2^{-O\left(n^{\frac{1}{3}}\right)}$.

The exact values of $M_{LP}(n,\theta)$ and $N_{LP}(n,k)$ are not known, and the relation of these derived quantities to $M(n,\theta)$ and $N(n,k)$ makes them legitimate subjects of research. In this paper we obtain a lower bound on $M_{LP}(n,\theta)$ and an upper bound on $N_{LP}(n,k)$. This sets limits on how good the bounds on codes and designs obtained through linear programming methods could be. We follow the approach in [11].

We prove:

Proposition 1.1: *For $n \geq 7$,* [1]

$$M_{LP}(n,\theta) \geq \Omega\left(\frac{1}{rn^{1/2}}\right)\left(\frac{1}{1-\delta^2}\right)^{\frac{n-4}{4}} \frac{P_r(1)}{\|P_r\|_2}, \tag{3}$$

where $\delta = \cos\theta$, $r := \max\left\{s \ : \ x_s \leq \delta - 2n^{-\frac{1}{2}}\right\}$, and x_s denotes the maximal root of P_s.

Proposition 1.2: *Let $\ell = k$ if k is even, and $\ell = k+1$ if k is odd. Then for $n \geq 6$,*

$$N_{LP}(n,k) \leq O\left(k\right)\left(\frac{1}{1-\rho^2}\right)^{\frac{n-2}{4}} \cdot \frac{P_\ell(1)}{\|P_\ell\|_2}, \tag{4}$$

where ρ is the maximal root of $P_\ell^{\frac{n-5}{2},\frac{n-5}{2}}$. [2]

Analyzing the asymptotic behaviour of the bounds leads to following corollaries.

Corollary 1.3:

$$\frac{1}{n}\log M_{LP}(n,\theta) \geq \frac{\log\frac{1}{\sin\theta}}{2} + \frac{\frac{1+\sin\theta}{2\sin\theta}\log\frac{1+\sin\theta}{2\sin\theta} - \frac{1-\sin\theta}{2\sin\theta}\log\frac{1-\sin\theta}{2\sin\theta}}{2} + o(1),$$

as n goes to infinity.

Corollary 1.4:

$$N_{LP}(n,k) \leq O\left(n^{-\frac{1}{2}} \cdot \left(\frac{\sqrt{2e}}{n}\right)^{n-3} \cdot k^{n-1}\right),$$

for n fixed and k going to infinity.

Combining this upper bound with Yudin's lower bound on $N_{LP}(n,k)$ (or with the lower bound of Delsarte, Goethals, and Seidel) we obtain

$$N_{LP}(n,k) = \Theta\left(k^{n-1}\right),$$

for n fixed and k going to infinity.

Note that the bound in (1.3) is, asymptotically, a geometric mean of the existential lower bound and the Kabatyansky-Levenshtein upper bound on $M(n,\theta)$. One encounters a similar phenomenon [11] in the context of LP-bounds for binary and constant weight binary codes.

The paper is organized as follows: in the next section we provide relevant information about Jacobi polynomials. Propositions 1.1 and 1.2 are proved in sections 3 and 4.

[1] No significant attempt has been made, either here or in proposition 1.2, to extend the claim to the cases $n = 3,4,5,6$.

[2] Observe that bounds (3) and (4) are, in a certain sense, up to polynomial factors, 'dual' to each other, as are linear programs (1) and (2).

2 Preliminaries

We will require some facts about Jacobi polynomials $P_s^{\alpha,\alpha}$. These facts are presented in this section.

Normalization [12]

$$P_s(1) = \left(\frac{n-1}{2}\right)_s, \tag{5}$$

where $(x)_s := x(x+1)...(x+s-1)$.

$$\|P_s\|_2^2 = \frac{2^{n-2}}{2s+n-2}\frac{\Gamma^2\left(r+\frac{n-1}{2}\right)}{r!(r+n-3)!}. \tag{6}$$

Asymptotics of the maximal root [9]
Let x_s be the maximal root of P_s. Then estimates in [9] (cor. 5.17, identity 5.35) give:

$$(2\sqrt{s}-1)\sqrt{\frac{s+n-4}{(2s+n-6)(2s+n-4)}} \leq x_s \leq 2\sqrt{s-1}\sqrt{\frac{s+n-4}{(2s+n-6)(2s+n-4)}}.$$

It follows that for any $s > 0$, and $n \geq 6$

$$\left| x_s - \frac{\sqrt{4s(s+n)}}{2s+n} \right| \leq \sqrt{\frac{2}{n}}. \tag{7}$$

It also follows that for any $s > 1$ and $n \geq 4$,

$$1 - x_s^2 \geq \frac{(n-4)^2}{(2s+n-4)^2}. \tag{8}$$

Lemma 2.1: *Assuming $n \geq 6$, for any $s \geq 0$: $w(t)P_s^2(t)$ is a decreasing function of t in the interval $\left[x_s + 2n^{-\frac{1}{2}}, 1\right]$.*

Proof: Let $t \in \left[x_s + 2n^{-\frac{1}{2}}, 1\right]$. By (7), $t \geq \frac{\sqrt{4s(s+n)}}{2s+n} + \frac{1}{2}n^{-\frac{1}{2}}$. It is not hard to check that this implies $(n-1)^2t^2 - 4(1-t^2)s(s+n-2) \geq 4t^2$.

Now we can follow the analysis of [1] for $\frac{P_s'(t)}{P_s(t)}$, obtaining:

$$\frac{P_s'(t)}{P_s(t)} < \frac{(n-1)t - \sqrt{(n-1)^2t^2 - 4(1-t^2)s(s+n-2)}}{2(1-t^2)} < \frac{(n-3)t}{2(1-t^2)}.$$

We conclude the proof of the lemma by computing $\frac{d}{dt}\ln(w(t)P_s^2(t)) = 2\frac{P_s'(t)}{P_s(t)} - \frac{(n-3)t}{1-t^2} < 0$, for $t \in \left[x_s + 2n^{-\frac{1}{2}}, 1\right]$. ∎

Corollary 2.2: *Assuming $n \geq 6$, for any $s \geq 0$: $w(t)P_s(t)$ is a decreasing function of t in the interval $\left[x_s + 2n^{-\frac{1}{2}}, 1\right]$.*

Lemma 2.3: *Assuming $n \geq 7$, for any $r > 0$ holds:*

$$\sum_{s=r}^{\infty} \frac{\|P_s\|_2}{P_s(1)} \leq O(r) \cdot \frac{\|P_r\|_2}{P_r(1)}.$$

Proof: We will assume n is odd, the proof for even n is similar. Set $a_s = \frac{\|P_s\|}{P_s(1)}$. By [12]

$$a_s^2 = \frac{\frac{2^{n-2}}{2s+n-2} \cdot \frac{(s+(n-3)/2)!(s+(n-3)/2)!}{s!(s+n-3)!}}{\binom{s+(n-3)/2}{s}^2} = \frac{2^{n-2}}{\binom{n-3}{(n-3)/2}} \cdot \frac{1}{2s+n-2} \cdot \frac{1}{\binom{s+n-3}{s}}.$$

Therefore $\frac{a_{s+1}^2}{a_s^2} = \frac{2s+n-2}{2s+n} \cdot \frac{s+1}{s+n-2} \leq \frac{s+1}{s+n-2}$, and for any $t \geq 0$,

$$\frac{a_{s+t}^2}{a_s^2} = \prod_{i=0}^{t-1} \frac{a_{s+i+1}^2}{a_{s+i}^2} \leq \frac{(s+1)_t}{(s+n-2)_t} = \frac{(s+1)_{n-3}}{(s+t+1)_{n-3}}.$$

It follows that

$$\sum_{s=r}^{\infty} a_s \leq a_r \cdot \sum_{t=0}^{\infty} \sqrt{\frac{(r+1)_{n-3}}{(r+t+1)_{n-3}}}.$$

The last sum, assuming $n \geq 7$, is at most $a_r \cdot \left[O(r) + O(r^2) \sum_{k=r}^{\infty} \frac{1}{k^2}\right] = O(r)a_r$. ∎

Remark 2.4: Observe that the ratio $\frac{\|P_s\|_2}{P_s(1)}$ decreases with s. ∎

Theorem 2.5: *[10] For all $x \in [-1, 1]$*

$$|P_s(x)| \leq O(\sqrt{n}) \frac{\|P_s\|_2}{(1-x^2)^{\frac{n-2}{4}}}. \tag{9}$$

Lemma 2.6: *Assuming $n \geq 3$, for any $t > 0$ the ratio $\frac{P_s(1)P_s(t)}{\|P_s\|_2^2}$ is increasing in s for even s such that the maximal root of the Jacobi polynomial $P_{s+2}^{\frac{n-5}{2}, \frac{n-5}{2}}$ does not exceed t.*

Proof: Let $b_s = \frac{P_s(1)P_s(t)}{\|P_s\|_2^2}$. We have to prove that $b_{s+2} \geq b_s$, which is equivalent to

$$\frac{P_{s+2}(1)}{P_s(1)} \cdot \frac{\|P_s\|_2^2}{\|P_{s+2}\|_2^2} \cdot P_{s+2}(t) \geq P_s(t). \tag{10}$$

It will be useful to renormalize and work with the ultraspherical polynomials $C_s = C_s^{(\frac{n-2}{2})}$, which are proportional to Jacobi polynomials $P_s^{\frac{n-3}{2},\frac{n-3}{2}}$:

$$C_s = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma(s+n-2)}{\Gamma(n-2)\Gamma\left(s+\frac{n-1}{2}\right)} \cdot P_s$$

Rewriting (10) for ultraspherical polynomials, and substituting the values of $P_i(1)$ and $\|P_i\|_2^2$, for $i = s, s+2$, we get the following inequality to prove:

$$(2s+n+2)C_{s+2}(t) \geq (2s+n-2)C_s(t).$$

Consider the following identity ([2], p. 178, (36)):

$$\frac{n-4}{2} \cdot (C_{s+2}(t) - C_s(t)) = \left(s + \frac{n-2}{2}\right) C_{s+2}^{(\frac{n-4}{2})}(t).$$

In the assumed range for s, $C_{s+2}^{(\frac{n-4}{2})}(t) \geq 0$. Therefore $C_{s+2}(t) \geq C_s(t)$. In order to complete the proof it is sufficient to show that $C_{s+2}(t) = C_s^{(\frac{n-2}{2})}(t) \geq 0$. This is indeed true, because by a theorem of Markov ([12], (6.21.3)), if $\lambda > \beta$, then the maximal root of $C_s^{(\lambda)}$ is smaller than that of $C_s^{(\beta)}$. ∎

3 A lower bound on $M_{LP}(n, \theta)$

Proof: of proposition 1.1
First,

$$\int_\delta^1 F(t)w(t)dt \geq \int_{-1}^1 F(t)w(t)dt = a_0 \int_{-1}^1 P_0^2(t)w(t)dt = \int_{-1}^1 w(t)dt.$$

Therefore, there exists $t_0 \in [\delta, 1]$ for which $F(t_0)w(t_0) \geq \frac{1}{1-\delta}\int_{-1}^1 w(t)dt$. Let $F = F_1 + F_2$, where $F_1 := \sum_{s=0}^r a_s P_s$.

We would like to show that either

$$F_2(t_0)w(t_0) \geq \frac{1}{2(1-\delta)} \int_{-1}^1 w(t)dt$$

or

$$\left|F_2(\delta)w(\delta)\right| \geq \frac{1}{2(1-\delta)} \int_{-1}^1 w(t)dt.$$

If $F_1(t_0)w(t_0) \leq \frac{1}{2(1-\delta)}\int_{-1}^1 w(t)dt$, then the first inequality holds. Otherwise, by corollary 2.2,

$$F_1(\delta)w(\delta) \geq F_1(t_0)w(t_0) > \frac{1}{2(1-\delta)} \int_{-1}^1 w(t)dt.$$

Since $F(\delta) \leq 0$, it must be that $F_2(\delta)w(\delta) < -\frac{1}{2(1-\delta)}\int_{-1}^1 w(t)dt$, implying the second inequality.

Let t_m be one of the two points t_0, δ, so that $|F_2(t_m)w(t_m)| \geq \frac{1}{2(1-\delta)} \int_{-1}^{1} w(t)dt$. Then, using (9)

$$\frac{1}{2(1-\delta)} \int_{-1}^{1} w(t)dt \leq \Big|F_2(t_m)w(t_m)\Big| \leq w(t_m) \cdot \sum_{s=r+1}^{m} a_s \Big|P_s(t_m)\Big| \leq$$

$$(1-t_m^2)^{\frac{n-4}{4}} \cdot \sum_{s=r+1}^{m} a_s \|P_s\|_2 \leq (1-\delta^2)^{\frac{n-4}{4}} \cdot \sum_{s=r+1}^{m} a_s \|P_s\|_2.$$

Since all the coefficients a_s are nonnegative, they are bounded from above: $a_s \leq \frac{F(1)}{P_s(1)}$. Therefore

$$F(1) \geq \frac{1}{2(1-\delta)} \int_{-1}^{1} w(t)dt \cdot \left(\frac{1}{1-\delta^2}\right)^{\frac{n-4}{4}} \cdot \frac{1}{\sum_{s=r+1}^{m} \frac{\|P_s\|_2}{P_s(1)}} \geq \Omega\left(\frac{1}{rn^{1/2}}\right)\left(\frac{1}{1-\delta^2}\right)^{\frac{n-2}{4}} \frac{P_r(1)}{\|P_r\|_2}.$$

The last inequality uses lemma 2.3 and a simple fact: $\int_{-1}^{1} w(t)dt = \Theta\left(n^{-\frac{1}{2}}\right)$. ∎

Proof: of corollary 1.3

The main step is to estimate r. From (7), $\left|r - \frac{\frac{1}{1-\delta^2}-1}{2} \cdot n\right| \leq O\left(n^{\frac{1}{2}}\right)$. Now, the claim of the corollary is obtained using (5) and (6), and simplifying. ∎

4 An upper bound on $N_{LP}(n,k)$

Proof: of proposition 1.2

First, we may, without loss of generality, assume that F is symmetric around zero. Indeed, if F is not symmetric, consider the symmetric function $G = \frac{F(t)+F(-t)}{F(1)+F(-1)}$. Clearly $G \geq 0$ on $[-1,1]$, $G(1) = 1$, and in the expansion $G = \sum_{s=0}^{\infty} b_s P_s$, the coefficients b_s are nonpositive for $s \geq k$. Also $b_0 = \frac{2a_0}{F(1)+F(-1)} \leq 2a_0$, so it is sufficient to provide a lower bound for b_0.

This said, we assume that initially F is symmetric. This, in particular, implies that $a_s = 0$ for all odd s.

To make this proof as similar as possible, up to a 'duality', to the proof of proposition 1.1, we introduce two definitions: Let $A : \mathbf{N} \rightarrow \mathbf{R}$ be defined by $A(s) = a_s$. Then

$$A(s) = \frac{\int_{-1}^{1} F(t)P_s(t)w(t)dt}{\|P_s\|_2^2} = \int_{-1}^{1} F(t)\alpha_t(s)dt,$$

where

$$\alpha_t(s) = \frac{P_s(t)w(t)}{\|P_s\|_2^2}$$

is 'dual' to $P_s(t)$.

Now, $\sum_{s=0}^{k-1} P_s(1)A(s) \geq \sum_{s=0}^{m} P_s(1)A(s) = F(1) = 1$. Therefore, there exists an index $s_0 \in [0, k-1]$ such that $P_{s_0}(1)A(s_0) \geq \frac{1}{k}$.

Write $A = A_1 + A_2$, where $A_2(s) := \int_{-\rho}^{\rho} F(t)\alpha_t(s)dt$. Let $\ell = k$ if k is even, and $\ell = k+1$ if k is odd. We would like to show that either

$$P_{s_0}(1)A_2(s_0) \geq \frac{1}{2k},$$

or

$$P_\ell(1)\Big|A_2(\ell)\Big| \geq \frac{1}{2k}.$$

If $P_{s_0}(1)A_1(s_0) \leq \frac{1}{2k}$, then the first inequality is satisfied. Otherwise, observe that, by lemma 2.6, for every $t \in [\rho, 1]$ we have $P_\ell(1)\alpha_t(\ell) \geq P_{s_0}(1)\alpha_t(s_0)$. This is also true for all $t \in [-1, -\rho]$, since s_0, ℓ are even and consequently P_{s_0}, P_ℓ are symmetric around 0. Therefore

$$P_\ell(1)A_1(\ell) = P_\ell(1)\int_{[-1,\rho]\cup[\rho,1]} F(t)\alpha_t(\ell)dt \geq P_{s_0}(1)\int_{[-1,\rho]\cup[\rho,1]} F(t)\alpha_t(s_0)dt > \frac{1}{2k}.$$

Since $A(\ell) \leq 0$, it must be that $P_\ell(1)|A_2(\ell)| \geq P_\ell(1)A_1(\ell) > \frac{1}{2k}$, and the second inequality holds.

Let s be one of the two indices s_0, ℓ, so that $P_s(1)|A_2(s)| \geq \frac{1}{2k}$. Then

$$\frac{1}{2k} \leq P_s(1)\Big|A_2(s)\Big| = P_s(1)\cdot\left|\int_{-\rho}^{\rho} F(t)\alpha_t(s)dt\right| = \frac{P_s(1)}{\|P_s\|_2^2}\cdot\left|\int_{-\rho}^{\rho} F(t)P_s(t)w(t)dt\right|$$

$$\leq \frac{P_s(1)}{\|P_s\|_2^2}\cdot\max_{t\in[-\rho,\rho]}\Big|P_s(t)\Big|\int_{-1}^{1} F(t)w(t)dt = A(0)\cdot\int_{-1}^{1} w(t)dt\cdot\frac{P_s(1)}{\|P_s\|_2^2}\cdot\max_{t\in[-\rho,\rho]}\Big|P_s(t)\Big| \leq$$

$$A(0)\cdot\Omega(1)\cdot\left(\frac{1}{1-\rho^2}\right)^{\frac{n-2}{4}}\cdot\frac{P_s(1)}{\|P_s\|_2}.$$

The last inequality uses (9).

Therefore,

$$\frac{1}{a_0} = \frac{1}{A(0)} \leq O(k)\cdot\left(\frac{1}{1-\rho^2}\right)^{\frac{n-2}{4}}\cdot\frac{P_s(1)}{\|P_s\|_2} \leq O(k)\cdot\left(\frac{1}{1-\rho^2}\right)^{\frac{n-2}{4}}\cdot\frac{P_\ell(1)}{\|P_\ell\|_2},$$

using the fact that the fraction $\frac{P_s(1)}{\|P_s\|_2}$ is increasing in s (see remark 2.4). ∎

Proof: of corollary 1.4
The claim of the corollary is obtained using (5), (6), and (8), and simplifying. ∎

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