A Brief Introduction to First- and Second-Generation Wavelets

An auxiliary material for the ACM SIGGRAPH 2009 paper:
Edge-Avoiding Wavelets and their Applications

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In this additional note to [Fattal 2009] we briefly review the fundamentals of multi-resolution analysis (MRA) and wavelet bases and transforms. The reader is referred to [Mallat 1999] or [Burrus et al. 1998] for the complete derivation of wavelets theory.

Multi-Resolution Analysis. MRA consists of an hierarchy of linear spaces $\{V^j\}_{j=0}^J$ called *approximation spaces*, where $V^{j+1} \subset V^j$ and V^0 is the finest resolution. The difference spaces $W^{j+1} = V^j \setminus V^{j+1}$ are called the *detail spaces* as they contain the detail needed to refine one approximation level to the next. A wavelet basis for a MRA is constructed by two functions, the *scaling function* $\phi(t)$ and the *wavelet function* $\psi(t)$. The translates of the scaling function dilated by 2^j, $\phi_{j,n}(t) = \phi(t/2^j - n)$, span V^j, and the translates of the wavelet function, dilated by 2^j, $\psi_{j,n}(t) = \psi(t/2^j - k)$, span W^j.

In finite spaces, $\dim V^j = N/2^j, j = 0..J$ and $\dim W^j = N/2^j, j = 1..J$, where usually $J \propto \log_2 N$, every function $f \in V^j$ has a representation $a^j[n], n = 0...N/2^j - 1$, in terms of $\phi_{j,n}(t)$, i.e.,

$$f(t) = \sum_{n=0}^{N/2^j} a^j[n]\phi_{j,n}(t), \tag{1}$$

or alternatively since $V^j = V^{j+1} \oplus W^{j+1}$, f can be expressed by

$$f(t) = \sum_{n=0}^{N/2^{j+1}} a^{j+1}[n]\phi_{j+1,n}(t) + \sum_{n=0}^{N/2^{j+1}} d^{j+1}[n]\psi_{j+1,n}(t), \tag{2}$$

where the first sum expresses f's components in V^{j+1}, i.e., its coarser approximation, and the the second sum expresses its components in W^{j+1}, the detail in V^j missing from V^{j+1}. The linear combination coefficients $a^j[n]$ are called *approximation coefficients* and the $d^j[n]$ are called *detail coefficients*.

The *wavelet transform* is the mapping between

$$a^j[n], n = 0...N/2^j \longleftrightarrow a^{j+1}[n], d^{j+1}[n], n = 0...N/2^{j+1}. \tag{3}$$

Fast Wavelet Transform. Since $\phi(t/2^{j+1}) \in V^{j+1} \subset V^j$ and $\psi(t/2^{j+1}) \in W^{j+1} \subset V^j$ they can both be expressed by means of $\phi_{j,n}$, i.e.,

$$\phi_{j+1,k}(t) = \sqrt{2}\sum_{n=0}^{N-1} h[n]\phi_{j,n+2k}(t), \tag{4}$$

$$\psi_{j+1,k}(t) = \sqrt{2}\sum_{n=0}^{N-1} g[n]\phi_{j,n+2k}(t).$$

The filters $h[n]$ and $g[n]$ are known as the *conjugate mirror filters* (CMF) and are the main ingredient behind the fast cascade implementation of the wavelet transform. It can be shown that for orthogonal wavelets [Mallat 1999; Burrus et al. 1998] the forward wavelet transform, from level j to $j+1$, is given by

$$a^{j+1}[n] = \big(a^j * \bar{h}\big)[2n], \quad \text{and} \quad d^{j+1}[n] = \big(a^j * \bar{g}\big)[2n], \tag{5}$$

where $*$ indicates a discrete (circular) convolution and $\bar{h}[n] = h[-n]$ and $\bar{g}[n] = g[-n]$ for real h and g. In this case, the inverse wavelet transform is computed by

$$a^j[n] = \big(\check{a}^{j+1} * h\big)[n] + \big(\check{d}^{j+1} * g\big)[n], \tag{6}$$

where $\check{a}[n] = a[n/2]$ and $\check{d}[n] = d[n/2]$ for even n, and $\check{a}[n] = 0$ and $\check{d}[n] = 0$ otherwise. The filters h and g must obey certain conditions described in [Mallat 1999; Burrus et al. 1998] related to the orthogonality of ϕ and ψ and a requirement that h is a low-pass and g is a high-pass filters. Therefore, intuitively speaking, $a^{j+1}[n]$ contains information for a *smooth* approximation of $a^j[n]$ and $d^{j+1}[n]$ is the residual *finer-scale* correction. Biorthogonality is a relaxation of the orthogonality condition over the scaling and wavelet functions where *dual* scaling $\tilde{\phi}$ and wavelet $\tilde{\psi}$ functions are introduced and obey

$$\langle \phi_{j,n}, \tilde{\phi}_{j,k} \rangle = \delta[n-k], \qquad \langle \psi_{j,n}, \tilde{\psi}_{j,k} \rangle = \delta[n-k], \tag{7}$$

$$\langle \phi_{j,n}, \tilde{\psi}_{j,k} \rangle = 0, \qquad \langle \psi_{j,n}, \tilde{\phi}_{j,k} \rangle = 0, \tag{8}$$

where $\langle \cdot, \cdot \rangle$ is the usual dot-product in L^2 and $\delta[0] = 1$ and zero otherwise. Biorthogonality provides more freedom in the construction of the mirror filters g, h and the dual mirror filters $\tilde{h}, \tilde{g}$ (defined from an equation analogous to (4) for the dual scaling and wavelet functions). The fast wavelet transform for birothogonal wavelets is given by

$$a^{j+1}[n] = \big(a^j * \bar{h}\big)[2n], \quad \text{and} \quad d^{j+1}[n] = \big(a^j * \bar{g}\big)[2n], \tag{9}$$

and the inverse transform uses the dual mirror filters,

$$a^j[n] = \big(\check{a}^{j+1} * \tilde{h}\big)[n] + \big(\check{d}^{j+1} * \tilde{g}\big)[n]. \tag{10}$$

Wavelet bases, unlike Fourier modes, are localized in space providing spatial information on top of the spectral one. Nonetheless, wavelets-based MRA, like Fourier transform, are based on a uniform notion of smoothness across space. This follows from the use of a *single* pair of scaling and wavelet functions to define smoothness everywhere in space, and appears as the convolution operation in equations (5). Strictly speaking, this spatial invariance is true only at every scale of the MRA; wavelet transforms that subsample the coarser scale do not operate, as a whole, as translation-invariant operators from the prospects of the fine scale. However, our argument is that wherever smoothness is measured throughout space, it is measured the same.

Second-Generation Wavelets. Second-generation wavelets do not correspond to the translates and dilates of the same mother-wavelet function, and the same goes for the scaling functions. This

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means that both $\phi_{j,n}$ and $\psi_{j,n}$ are defined though a scaling equation, analogous to (4), using *spatially-varying* CMFs $h_{j,k}[n]$ and $g_{j,k}[n]$, as given by

$$\phi_{j,k}(t) = \sqrt{2}\sum_{n=0}^{N-1} h_{j,k}[n]\phi_{j,n+2k}(t), \tag{11}$$

$$\psi_{j,k}(t) = \sqrt{2}\sum_{n=0}^{N-1} g_{j,k}[n]\phi_{j,n+2k}(t).$$

Sweldens showed [1998] that given any set of biorthogonal CMF $h^{old}_{j,k}, g^{old}_{j,k}, \tilde{g}^{old}_{j,k}$, and $\tilde{h}^{old}_{j,k}$, which may be translation-invariant such as the Lazy wavelets, i.e., $h^{old}_{j,k}[n] = g^{old}_{j,k}[n] = \tilde{g}^{old}_{j,k}[n] = \tilde{h}^{old}_{j,k}[n] = \delta[n-k]$, one can obtain another biorthogonal family by two alterations

$$g_{j,k}[n] = g^{old}[n] - \sum_m h^{old}_{j,n+2m} l^p_{j,k}[m], \quad h[n] = h^{old}[n], \tag{12}$$

$$\tilde{h}_{j,k}[n] = \tilde{h}^{old}[n] + \sum_m \tilde{g}^{old}_{j,n+2mn} l^p_{j,k}[m], \quad \tilde{g}[n] = \tilde{g}^{old}[n], \tag{13}$$

which corresponds to the prediction step in the *lifting scheme*, with the prediction operator defined by l^p, and

$$h_{j,k}[n] = h^{old}[n] + \sum_m g^{old}_{j,n+2m} l^u_{j,k}[m], \quad g[n] = g^{old}[n], \tag{14}$$

$$\tilde{g}_{j,k}[n] = \tilde{g}^{old}[n] - \sum_m \tilde{h}^{old}_{j,n+2m} l^u_{j,k}[m], \quad \tilde{h}[n] = \tilde{h}^{old}[n], \tag{15}$$

corresponds to to the update step, with the update operator defined by l^u.

This schemes allows to improve and adapt the wavelet analysis and synthesis to the different particularities of the problem and do so in a *spatially-dependent* manner, through $l_{j,k}[n]$.

Rendering the Scaling and Wavelet Functions. In many applications, the finest level V^0 is interpreted as the original sampling grid of image pixels, meaning that $a^0[n]$ correspond to pixel values. Therefore, if we wish to 'render,' i.e., express $\phi_{j,n}$ and $\psi_{j,n}$ for $j \geq 1$ in terms of pixels, we should find their linear combination by $\phi_{0,n}$. This corresponds to defining an MRA where *all* the approximation and detail coefficients are set to zero except for the coefficients corresponding to the desired function $\phi_{j,n}$ or $\psi_{j,n}$ which is set to one, i.e., $a^j[n] = 1$ or $d^j[n] = 1$ respectively. Applying the inverse wavelet transform up to level zero, maps this representation to $a^0[n]$ as desired.

References

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