

1.4.1 Poisson Formula

$$\sum_{n=-\infty}^{\infty} e^{-inT\omega} = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right).$$

Proof: We'll denote the r.h.s. by $c(t)$ and notice that $\hat{c}(\omega)$ is a periodic function with a period of $2\pi/T$. Meaning that it will be enough to show that $\hat{c}(\omega) = \frac{2\pi}{T}\delta(\omega)$ for $\omega \in [-\pi/T, \pi/T]$. We'll use a test-function $\hat{\phi}(\omega)$ to show that

$$\langle \hat{c}, \hat{\phi} \rangle = \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \sum_{n=-N}^N e^{-i\omega n T} \hat{\phi}(\omega) d\omega = \frac{2\pi}{T} \hat{\phi}(\omega)(0).$$

The finite geometric sum can be written as

$$\begin{aligned} \sum_{n=-N}^N e^{-i\omega n T} &= e^{i\omega N T} \sum_{n=0}^{2N} e^{-i\omega n T} = \frac{e^{i\omega(N+1/2)T} (1 - e^{-i\omega(2N+1)T})}{e^{i\omega T/2} (1 - e^{-i\omega T})} = \\ &= \frac{e^{i\omega(N+1/2)T} - e^{-i\omega(N+1/2)T}}{e^{i\omega T/2} - e^{-i\omega T/2}} = \frac{\sin((N+1/2)\omega T)}{\sin(\omega T/2)}. \end{aligned}$$

($\sum_{i=0}^{n-1} a^i = \frac{1-a^n}{1-a}$). Thus,

$$\langle \hat{c}, \hat{\phi} \rangle = \lim_{N \rightarrow \infty} \frac{2\pi}{T} \int_{-\pi/T}^{\pi/T} \frac{\sin((N+1/2)\omega T)}{\pi\omega} \frac{\omega T/2}{\sin(\omega T/2)} \hat{\phi}(\omega) d\omega.$$

(the limited integration domain results from $\hat{\phi}$'s limited support). We define in $[-\pi/T, \pi/T]$

$$\hat{\psi}(\omega) = \hat{\phi}(\omega) \frac{\omega T/2}{\sin(\omega T/2)},$$

use the identity that $\int_{-T}^T e^{-i\omega t} dt = \frac{2\sin(T\omega)}{\omega}$, and get from Parseval

$$\langle \hat{c}, \hat{\phi} \rangle = \lim_{N \rightarrow \infty} \frac{2\pi}{T} \int_{-\infty}^{\infty} \frac{\sin((N+1/2)\omega T)}{\pi\omega} \hat{\psi}(\omega) d\omega = \lim_{N \rightarrow \infty} \frac{2\pi}{T} \int_{-(N+1/2)T}^{(N+1/2)T} \psi(t) dt.$$

(note that the 2π falls from the 2 in the sinc and π from the denominator, and replaced by the same from the Parseval) When n goes to infinity the integral equals to $\hat{\psi}(0)$ which, in turn, equals to $\hat{\phi}(0)$.