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Determining Compatibility Coefficients for Curve Enhancement Relaxation Processes

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***Abstract*—Relaxation labeling is a process that attempts to disambiguate probabilistic labelings of objects. Compatibility coefficients play an important role in the relaxation process. No explanation exists at present for their exact meaning, and no algorithm has been proposed to generate them. Some possible interpretations of these coefficients are presented, and algorithms are suggested to obtain them from the initial probabilistic labeling. Examples are given for the case where relaxation is used to disambiguate the detection of curves in a picture.**

I. Introduction

In many image processing tasks a classification of each point into one of several classes is desired. For example, in line detection, points are classified as being on a line having a certain direction or as not being on a line. However, classification processes such as this, based on local detection, do not usually give perfect results. The processes are sensitive to local noise and sometimes cannot determine the exact classification. A line detector, for example, can often find substantial responses in several directions at a given point. A method that has been used to improve the initial classification is relaxation labeling.

To apply relaxation labeling, we initially assign to each point the probabilities of its possible class memberships, based on local information. The relaxation labeling process uses knowledge of how labels interact locally to improve and disambiguate this prior classification. This process is described in Section II. This correspondence suggests several automatic methods of determining the mutual support coefficients used in relaxation according to different possible interpretations of these coefficients.

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II. The Relaxation Labeling Process

In this section we review some of the concepts involved in relaxation labeling. The subject is discussed in greater detail in [1]; for further references see [2].

The relaxation process involves a set of objects $A = \{a_1, a_2, \cdots, a_n\}$ and a set of labels (class names) $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$. For each object a_i we are given a set of local measurements, which are used as a basis for estimating the probabilities $P_i(\lambda)$ of object a_i having each label λ. These probabilities satisfy the condition

$$\sum_{\lambda \in \Lambda} P_i(\lambda) = 1, \qquad \text{for all } a_i \in A, \qquad 0 \leq P_i(\lambda) \leq 1. \tag{1}$$

The relaxation process is a parallel algorithm that updates the probabilities of labels. The probabilities are updated using a set of

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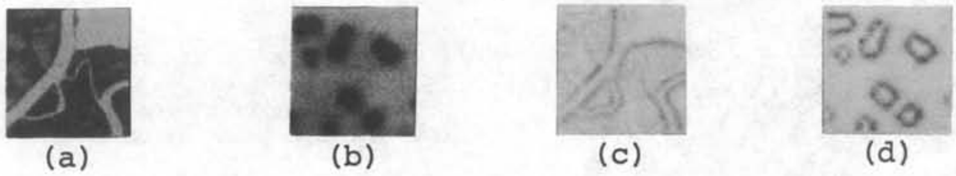

Fig. 1. Images used in experiments. (a), (b) Original images. (c), (d) Edge detector outputs.

given "compatibility coefficients" $r_{ij}(\lambda, \lambda')$, where $r_{ij}: \Lambda \times \Lambda \rightarrow [-1, 1]$ and:

a) if λ and λ' are compatible for objects a_i and a_j, respectively, then $r_{ij}(\lambda, \lambda') > 0$;
b) if λ and λ' are incompatible for a_i and a_j, respectively, then $r_{ij}(\lambda, \lambda') < 0$;
c) if neither labeling is constrained by the other, then $r_{ij}(\lambda, \lambda') = 0$;
d) the magnitude of r_{ij} represents the strength of the compatibility.

Methods for computing the r_{ij} are suggested in Sections IV and V.

We now discuss the probability updating rule. The updating factor for the estimate $P_i^k(\lambda)$ (at the kth iteration) is

$$q_i^k(\lambda) = \frac{1}{n} \sum_j \sum_{\lambda'} r_{ij}(\lambda, \lambda') P_j^k(\lambda') \quad (2)$$

where n is the number of objects. The new estimate of the probability of λ at a_i is

$$P_i^{(k+1)}(\lambda) = \frac{P_i^k(\lambda)[1 + q_i^k(\lambda)]}{\sum_{\lambda'} P_i^k(\lambda')[1 + q_i^k(\lambda')]}. \quad (3)$$

Thus each $P_i^k(\lambda)$ is multiplied by $[1 - q_i^k(\lambda)]$, and the values are normalized at each object to satisfy (1). The relaxation process is iterated until some termination criterion is met.

III. Curve Enhancement Using Relaxation

Since the domain from which the examples in this report are drawn is curve enhancement, this application is briefly described in this section. A detailed discussion appears in [3]. Line detectors are applied to the picture, and at each point probabilities are assigned to nine labels: lines in eight possible directions and a "no line" label.

Two methods of line detection can be used: linear and nonlinear. The response of a linear line detector for vertical lines, given the configuration of points

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

will be $2(b + e + h) - (a + c + d + f + g + i)$. The response of a nonlinear detector for vertical lines will be

$$\begin{cases} 2(b + e + h) - (a + c + d + f + g + i), & \\ \qquad \text{if } a < b > c, \quad d < e > f, \quad g < h > i & \\ 0, & \\ \qquad \text{otherwise.} & \end{cases}$$

The initial probabilistic labeling is obtained from the line detector output by normalization (see [3]).

The coefficients used in the probability updating rule will be discussed below. They are computed only for neighboring pairs of points; in other words, they are assumed to be zero for nonneighboring point pairs.

The following simplifications have been made in the relaxation process described in Section II. The probability of a label whose current estimate is greater than 0.9 and which gets maximum support is increased to 1, and that point is never considered again for updating. Also, the $q_i(\lambda)$ of (2) are not divided by n (when this does not violate the condition $q_i(\lambda) > -1$), in order to increase the effect of each iteration on the $P_i(\lambda)$.

Input images for our curve enhancement experiments were obtained by applying an edge detection operator (based on differences of two-by-two averages) to the two pictures shown in Fig. 1(a), (b), yielding the outline pictures shown in Fig. 1(c), (d).

IV. Compatibility Coefficients as Correlations

For brevity, let ρ_{ij} denote the compatibility between the points (x, y) and $(x + i, y + j)$—e.g., $\rho_{10}(\lambda, \lambda')$ is the compatibility between label λ at a point and label λ' at its right-hand neighbor.

One possible interpretation of the compatibilities is in terms of statistical correlation, since correlation has properties a)–d) listed in Section II. Estimates of the correlation coefficients derived from analyzing the initial labeling are

$$R_{ij}(\lambda, \lambda') = \frac{\sum_{(x,y)} [P_{(x,y)}(\lambda) - \bar{P}(\lambda)][P_{(x+i,y+j)}(\lambda') - \bar{P}(\lambda')]}{\sigma(\lambda)\sigma(\lambda')}. \quad (4)$$

Here $P_{(a,b)}(\lambda)$ is the initial estimate of the probability of labeling point (a, b) with λ, $\bar{P}(\lambda)$ is the average of $P_{(x,y)}(\lambda)$ for all points (x, y), and $\sigma(\lambda)$ is the standard deviation of $P_{(x,y)}(\lambda)$.

To test these R_{ij} in a line enhancement relaxation process, nonlinear line detection operators (see Section III) in eight orientations were applied to the picture in Fig. 1(c), and the response strengths were used, as in [3], to compute initial estimates of the probabilities of the nine labels (eight line orientations and "no line"). Fig. 2(a) is a symbolic representation of the most probable label at each point; the "no line" labels are represented by solid 3×3 squares with gray level proportional to the "no line" probability (0 = black, 1 = white), while the line labels are represented by three-point line segments in the appropriate orientations with gray levels "negatively proportional" to the line probability (0 = white, 1 = black). It can be seen that the line detections are somewhat noisy.

Table I shows correlation coefficients derived from Fig. 2(a) using (4), and Fig. 2(b)–(d) show iterations 2, 6, and 12 of the relaxation process using these coefficients, applied to Fig. 2(a). The results are very poor. This is because when one label dominates the picture, as the "no line" label does in our case, its correlation coefficients with all other labels are high. Thus in the relaxation updating, the "no line" label gets most of the support, and after a few iterations almost all the points have this label.

The effect of dominance among labels can be alleviated by weighting the R's by the probabilities that the corresponding labels do not occur. This greatly reduces the values of the coefficients involving dominant labels, but will only slightly reduce the values of the coefficients involving rare labels. The new coefficients are

$$R_{ij}^*(\lambda, \lambda') = [1 - \bar{P}(\lambda)][1 - \bar{P}(\lambda')] \cdot R_{ij}(\lambda, \lambda') \quad (5)$$

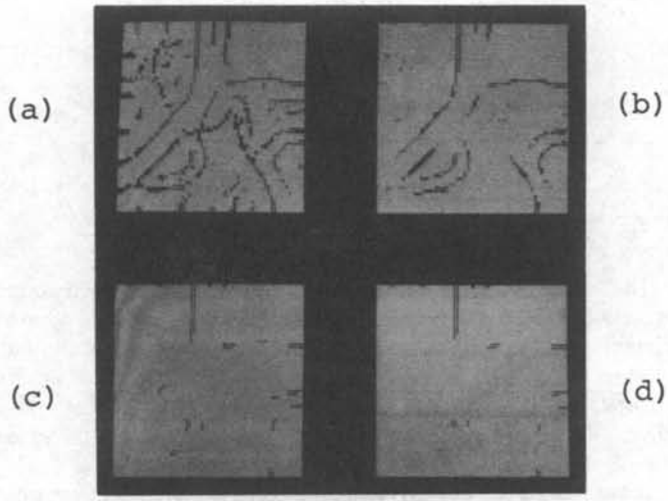

Fig. 2. Iterations 0, 2, 6, 12 of relaxation process using unweighted correlation coefficients derived from Fig. 2(a). (For explanation of symbols in these figures, see text.)

where R_{ij} is defined by (4).[1] These coefficients have the property that they support rare labels at points having no evidence from their neighborhoods, since the self-support coefficients (the R^*_{00}) are the highest for the rare labels. In case this effect is undesirable, one might ignore the self-support in the relaxation.

The coefficients defined by (5) for the case of Fig. 2(a) are shown in Table II. Fig. 3 shows iterations 0, 2, 6, and 12 of the relaxation process using these coefficients, and Fig. 4 is analogous except that the self-support coefficients are ignored. (Figs. 3(a) and 4(a) are the same as Fig. 2(a).) The results are extremely similar and are much better than those in Fig. 2. The ambiguity has been greatly reduced, and the curves have survived the process. Note that the surviving curve points are, for the most part, just the points whose most probable initial label was a line label; the results do not differ greatly from a maximum-likelihood classification of the points based on the initial probability estimates.

It should be noted that the ideal correlation coefficients should be symmetric, i.e., pairs of neighbors on opposite sides of a point should yield the same sets of coefficients (transposed). In the estimated coefficients of Tables I and II, this is almost exactly true. Any discrepancy is due partly to computational error and partly to picture border effects. Analogous remarks apply to the tables of mutual information coefficients (Table III, ff.).

V. Compatibility Coefficients as "Mutual Information"

A different approach to defining compatibility coefficients is based on the mutual information of the labels at neighboring points. This approach too satisfies our intuitive ideas about the nature of the coefficients: If two labels have a high positive correlation, we can expect them to have a high mutual information, and vice versa.

We estimate the probability of any point having the label λ by

$$\bar{P}(\lambda) = \frac{1}{n} \sum_{(x,y)} P_{(x,y)}(\lambda) \tag{6}$$

[1] The choice of weighting function in (5) was somewhat arbitrary. For example, we could have chosen to define the weight to be $\sqrt{[1-\bar{P}(\lambda)][1-\bar{P}(\lambda')]}$ rather than $[1-\bar{P}(\lambda)][1-\bar{P}(\lambda')]$, i.e., the geometric mean rather than the product.

TABLE I
Correlation Coefficients Derived from Fig. 2(a)

	0.16	-0.19	0.03	0.04	0.02	-0.07	-0.06	-0.10	-0.02
	-0.12	0.26	-0.02	-0.02	-0.01	-0.03	-0.00	-0.01	-0.02
	0.03	-0.02	0.00	-0.01	0.00	-0.02	-0.01	-0.01	-0.01
	0.04	-0.02	-0.01	-0.02	0.02	-0.02	-0.01	-0.01	-0.01
a)	0.03	-0.02	-0.01	-0.01	-0.01	-0.02	-0.01	-0.01	-0.01
	-0.08	-0.03	-0.01	-0.02	-0.02	0.21	0.08	-0.01	-0.02
	-0.03	0.01	-0.01	-0.01	0.03	-0.01	0.19	-0.00	-0.01
	-0.06	-0.01	-0.01	-0.01	-0.01	0.00	-0.00	0.43	-0.01
	-0.18	0.19	-0.01	-0.01	-0.01	-0.02	-0.01	0.15	0.16
	0.49	-0.50	-0.09	-0.08	-0.01	-0.09	-0.04	-0.01	-0.20
	-0.53	0.83	0.03	-0.02	-0.02	-0.03	-0.01	-0.01	0.13
	-0.16	0.10	0.32	0.07	-0.00	-0.02	-0.01	0.01	0.00
	-0.08	-0.02	-0.01	0.29	0.05	-0.02	-0.01	-0.01	0.02
b)	0.02	-0.02	-0.01	-0.01	0.03	-0.02	-0.01	-0.01	-0.01
	-0.11	-0.03	-0.01	-0.02	-0.02	0.25	0.06	-0.01	-0.02
	-0.03	-0.01	0.00	0.02	0.07	-0.01	0.13	-0.00	-0.01
	-0.06	-0.01	-0.01	-0.01	-0.01	0.00	-0.00	0.19	0.13
	-0.08	-0.02	0.00	-0.01	-0.01	-0.02	-0.01	-0.01	0.35
	0.29	-0.13	-0.14	-0.25	-0.13	-0.11	-0.01	0.02	-0.03
	-0.18	0.27	0.08	-0.02	-0.02	-0.03	-0.02	-0.01	0.08
	-0.04	-0.02	0.21	0.07	-0.00	-0.02	-0.01	-0.01	-0.01
	-0.25	-0.02	0.11	0.78	0.03	-0.02	-0.01	-0.01	-0.01
c)	-0.06	-0.00	-0.01	0.02	0.29	-0.02	-0.01	-0.01	-0.01
	-0.15	-0.03	-0.02	-0.02	0.11	0.28	0.06	-0.01	-0.02
	-0.00	-0.00	0.00	0.02	-0.00	0.01	-0.01	-0.00	-0.01
	0.01	-0.01	0.03	-0.01	-0.01	-0.01	-0.00	-0.00	-0.01
	0.01	-0.02	0.04	0.01	-0.01	-0.02	-0.01	-0.01	0.00
	0.40	-0.17	0.02	-0.06	-0.13	-0.43	-0.04	-0.07	0.00
	-0.12	0.26	-0.02	-0.02	-0.02	-0.03	-0.01	-0.01	-0.02
	-0.02	-0.02	0.06	-0.01	0.01	-0.02	-0.01	-0.01	0.08
	-0.06	-0.02	-0.01	0.29	0.00	-0.02	-0.01	-0.01	0.02
d)	-0.09	-0.02	-0.01	-0.01	0.36	0.05	-0.01	-0.01	-0.01
	-0.42	-0.03	-0.02	-0.02	0.10	0.81	-0.01	-0.01	-0.02
	-0.10	-0.02	-0.01	0.03	-0.01	0.10	0.36	-0.00	-0.01
	0.01	-0.01	-0.01	-0.01	-0.01	0.00	-0.00	0.05	-0.01
	-0.11	0.15	-0.01	-0.01	-0.01	-0.02	-0.01	0.25	-0.01
	0.99	-0.58	-0.26	-0.29	-0.26	-0.51	-0.14	-0.14	-0.29
	-0.58	0.99	0.02	-0.02	-0.02	-0.03	-0.01	-0.01	-0.02
	-0.26	0.02	1.00	0.03	-0.00	-0.02	-0.01	-0.01	-0.01
	-0.29	-0.02	0.03	1.00	0.00	-0.02	-0.01	-0.01	-0.01
e)	-0.26	-0.02	-0.00	0.00	1.00	0.04	-0.01	-0.01	-0.01
	-0.51	-0.03	-0.02	-0.02	0.04	1.00	-0.01	-0.01	-0.02
	-0.14	-0.01	-0.01	-0.01	-0.01	-0.01	1.00	-0.00	-0.01
	-0.14	-0.01	-0.01	-0.01	-0.01	-0.01	-0.00	1.00	-0.01
	-0.29	-0.02	-0.01	-0.01	-0.01	-0.02	-0.01	-0.01	1.00
	0.41	-0.12	-0.02	-0.06	-0.09	-0.44	-0.10	0.01	-0.11
	-0.17	0.26	-0.02	-0.02	-0.02	-0.03	-0.02	-0.01	0.15
	0.02	-0.02	0.06	-0.01	-0.01	-0.02	-0.01	-0.01	-0.01
	-0.06	-0.02	-0.01	0.29	-0.01	-0.02	0.03	-0.01	-0.01
f)	-0.13	-0.02	0.01	0.00	0.36	0.10	-0.01	-0.01	-0.01
	-0.45	-0.03	-0.02	-0.02	0.05	0.85	0.10	0.00	-0.02
	-0.04	-0.01	-0.01	-0.01	-0.01	-0.01	0.36	-0.00	-0.01
	-0.07	-0.01	-0.01	-0.01	-0.01	-0.01	-0.00	0.05	0.25
	0.00	-0.02	0.08	0.02	-0.01	-0.02	-0.01	-0.01	-0.01
	0.29	-0.18	-0.04	-0.25	-0.06	-0.14	-0.00	0.01	0.01
	-0.13	0.27	-0.02	-0.02	-0.00	-0.03	-0.00	-0.01	-0.02
	-0.14	0.08	0.21	0.11	-0.01	-0.02	0.00	0.03	0.04
	-0.25	-0.02	0.07	0.78	0.02	-0.02	0.02	-0.01	0.01
g)	-0.13	-0.02	-0.00	0.03	0.29	0.11	-0.00	-0.01	-0.01
	-0.10	-0.03	-0.02	-0.02	-0.02	0.27	0.01	-0.01	-0.02
	-0.01	-0.02	-0.01	-0.01	-0.01	0.06	-0.01	-0.00	-0.01
	0.02	-0.01	-0.01	-0.01	-0.01	-0.01	-0.00	-0.00	-0.01
	-0.03	0.08	-0.01	-0.01	-0.01	-0.02	-0.01	-0.01	0.00
	0.48	-0.52	-0.16	-0.08	0.02	-0.11	-0.03	-0.06	-0.08
	-0.49	0.82	0.10	-0.02	-0.02	-0.03	-0.01	-0.01	-0.02
	-0.09	0.03	0.32	-0.01	-0.01	-0.01	0.00	-0.01	0.00
	-0.08	-0.02	0.07	0.29	-0.01	-0.02	0.02	-0.01	-0.01
h)	-0.01	-0.02	-0.00	0.05	0.03	-0.02	0.07	-0.01	-0.01
	-0.09	-0.03	-0.02	-0.02	-0.02	0.25	-0.01	0.00	-0.02
	-0.04	-0.01	-0.01	-0.01	-0.01	0.06	0.13	-0.00	-0.01
	-0.01	-0.01	0.01	-0.01	-0.01	-0.01	-0.00	0.19	-0.01
	-0.20	0.13	0.00	0.02	-0.01	-0.02	-0.01	0.13	0.35
	0.16	-0.12	0.03	0.04	0.03	-0.09	-0.03	-0.06	-0.18
	-0.19	0.25	-0.02	-0.02	-0.02	-0.03	0.01	-0.01	0.19
	0.03	-0.02	0.00	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01
	0.04	-0.02	-0.01	-0.02	-0.01	-0.02	-0.01	-0.01	-0.01
i)	0.02	-0.01	0.00	0.02	-0.01	-0.02	0.03	-0.01	-0.01
	-0.08	-0.03	-0.02	-0.02	-0.02	0.22	-0.01	0.00	-0.02
	-0.06	-0.00	-0.01	-0.01	-0.01	0.08	0.19	-0.00	-0.01
	-0.10	-0.01	-0.01	-0.01	-0.01	-0.01	-0.00	0.43	0.15
	-0.02	-0.02	-0.01	-0.01	-0.01	-0.02	-0.01	-0.01	0.16

The nine parts (a-i) correspond to the nine neighbors of point (e) in the following order:

```
a b c
d e f
g h i
```

In each part, the first row and first column correspond to no-line probabilities; the remaining rows and columns correspond to slopes measured clockwise from the vertical in steps of $22\frac{1}{2}°$.

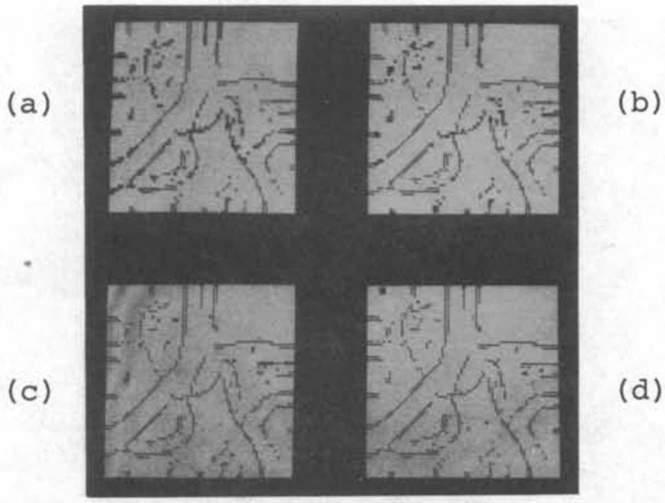

Fig. 3. Analogous to Fig. 2, but using weighted correlation coefficients.

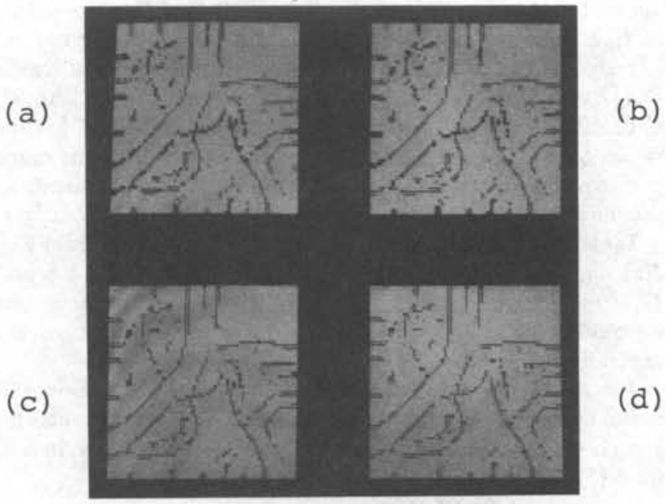

Fig. 4. Same as Fig. 3, but ignoring the self-support coefficients.

and the joint probability of a pair of points having labels λ and λ' by

$$P_{ij}(\lambda, \lambda') \equiv \frac{1}{n} \sum_{(x,y)} P_{(x,y)}(\lambda) P_{(x+i,y+j)}(\lambda') \tag{7}$$

where n is the number of points and $P_{(x,y)}(\lambda)$ is the initial estimate of the probability of point (x, y) having the label λ. We can now estimate the conditional probability that (x, y) is labeled λ given that $(x + i, y + j)$ is labeled λ' by

$$P_{ij}(\lambda \mid \lambda') \equiv \frac{P_{ij}(\lambda, \lambda')}{\bar{P}(\lambda')} = \frac{\sum_{(x,y)} P_{(x,y)}(\lambda) P_{(x+i,y+j)}(\lambda')}{\sum_{(x,y)} P_{(x,y)}(\lambda')}. \tag{8}$$

For any event A whose probability of occurrence is $P(A)$, the amount of information we receive as a result of being told that A has occurred is defined by

$$I(A) = -\log P(A). \tag{9}$$

This information is zero when $P(A) = 1$, since we knew already that A will occur, and approaches infinity when $P(A)$ approaches zero. In the same manner, the conditional information that we

TABLE II
WEIGHTED CORRELATION COEFFICIENTS DERIVED FROM FIG. 2(a)

a)	0.00	-0.01	0.00	0.00	0.00	-0.00	-0.00	-0.01	-0.00
	-0.01	0.25	-0.02	-0.02	-0.01	-0.03	-0.00	-0.01	-0.02
	0.00	-0.02	0.00	-0.01	0.00	-0.02	-0.01	-0.01	-0.01
	0.00	-0.02	-0.01	-0.02	0.02	[illegible]	-0.01	-0.01	-0.01
	0.00	-0.01	-0.01	-0.01	-0.01	[illegible]	-0.01	-0.01	-0.01
	-0.01	-0.03	-0.01	-0.02	-0.02	[illegible]	0.08	-0.01	-0.02
	-0.00	0.01	-0.01	-0.01	0.03	-0.01	0.19	-0.00	-0.01
	-0.00	-0.01	-0.01	-0.01	-0.01	0.00	-0.00	0.42	-0.01
	-0.01	0.19	-0.01	-0.01	-0.01	-0.02	-0.01	0.15	0.16
b)	0.00	-0.03	-0.01	-0.01	-0.00	-0.01	-0.00	-0.00	-0.01
	-0.04	0.79	0.03	-0.02	-0.02	-0.03	-0.01	-0.01	0.12
	-0.01	0.10	0.32	0.07	-0.00	-0.02	-0.01	0.01	0.00
	-0.01	-0.02	-0.01	0.29	0.05	-0.02	-0.01	-0.01	0.02
	0.00	-0.02	-0.01	-0.01	0.03	-0.02	-0.01	-0.01	-0.01
	-0.01	-0.03	-0.01	-0.02	-0.02	0.24	0.06	-0.01	-0.02
	-0.00	-0.01	0.00	0.02	0.07	-0.01	0.13	-0.00	-0.01
	-0.00	-0.01	-0.01	-0.01	-0.01	0.00	-0.00	0.19	0.13
	-0.01	-0.02	0.00	-0.01	-0.01	-0.02	-0.01	-0.01	0.34
c)	0.00	-0.01	-0.01	-0.02	-0.01	-0.01	-0.00	0.00	-0.00
	-0.01	0.26	0.08	-0.02	-0.02	-0.03	-0.02	-0.01	0.08
	-0.00	-0.02	0.21	0.06	-0.00	-0.02	-0.01	-0.01	-0.01
	-0.02	-0.02	0.10	0.76	0.03	-0.02	-0.01	-0.01	-0.01
	-0.00	-0.00	-0.01	0.02	0.29	-0.02	-0.01	-0.01	-0.01
	-0.01	-0.03	-0.02	-0.02	0.11	0.27	0.06	-0.01	-0.02
	-0.00	-0.00	0.00	0.02	-0.00	0.00	-0.01	-0.00	-0.01
	0.00	-0.01	0.03	-0.01	-0.01	-0.01	-0.00	-0.00	-0.01
	0.00	-0.02	0.04	0.01	-0.01	-0.02	-0.01	-0.01	0.00
d)	0.00	-0.01	0.00	-0.00	-0.01	-0.03	-0.00	-0.01	0.00
	-0.01	0.25	-0.02	-0.02	-0.02	-0.03	-0.01	-0.01	-0.02
	-0.00	-0.02	0.06	-0.01	0.01	-0.02	-0.01	-0.01	0.08
	-0.00	-0.02	-0.01	0.28	0.00	-0.02	-0.01	-0.01	0.02
	-0.01	-0.02	-0.01	-0.01	0.35	0.04	-0.01	-0.01	-0.01
	-0.03	-0.03	-0.02	-0.02	0.09	0.78	-0.01	-0.01	-0.02
	-0.01	-0.02	-0.01	0.03	-0.01	0.10	0.36	-0.00	-0.01
	0.00	-0.01	-0.01	-0.01	-0.01	0.00	-0.00	0.05	-0.01
	-0.01	0.14	-0.01	-0.01	-0.01	-0.02	-0.01	0.25	-0.01
e)	0.00	-0.04	-0.02	-0.02	-0.02	-0.03	-0.01	-0.01	-0.02
	-0.04	0.94	0.02	-0.02	-0.02	-0.03	-0.01	-0.01	-0.02
	-0.02	0.02	0.99	0.03	-0.00	-0.02	-0.01	-0.01	-0.01
	-0.02	-0.02	0.03	0.98	0.00	-0.02	-0.01	-0.01	-0.01
	-0.02	-0.02	-0.00	0.00	0.99	0.03	-0.01	-0.01	-0.01
	-0.03	-0.03	-0.02	-0.02	0.03	0.96	-0.01	-0.01	-0.02
	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	0.99	-0.00	-0.01
	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	-0.00	1.00	-0.01
	-0.02	-0.02	-0.01	-0.01	-0.01	-0.02	-0.01	-0.01	0.99
f)	0.00	-0.01	-0.00	-0.00	-0.01	-0.03	-0.01	0.00	-0.01
	-0.01	0.25	-0.02	-0.02	-0.02	-0.03	-0.02	-0.01	0.14
	0.00	-0.02	0.06	-0.01	-0.01	-0.02	-0.01	-0.01	-0.01
	-0.00	-0.02	-0.01	0.28	-0.01	-0.02	0.03	-0.01	-0.01
	-0.01	-0.02	0.01	0.00	0.35	0.09	-0.01	-0.01	-0.01
	-0.03	-0.03	-0.02	-0.02	0.04	0.82	0.10	0.00	-0.02
	-0.00	-0.01	-0.01	-0.01	-0.01	-0.01	0.36	-0.00	-0.01
	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	-0.00	0.05	0.25
	0.00	-0.02	0.08	0.02	-0.01	-0.02	-0.01	-0.01	-0.01
g)	0.00	-0.01	-0.00	-0.02	-0.00	-0.01	-0.00	0.00	0.00
	-0.01	0.26	-0.02	-0.02	-0.00	-0.03	-0.00	-0.01	-0.02
	-0.01	0.08	0.21	0.10	-0.01	-0.02	0.00	0.03	0.04
	-0.02	-0.02	0.06	0.76	0.02	-0.02	0.02	-0.01	0.01
	-0.01	-0.02	-0.00	0.03	0.29	0.11	-0.00	-0.01	-0.01
	-0.01	-0.03	-0.02	-0.02	-0.02	0.26	0.00	-0.01	-0.02
	-0.00	-0.02	-0.01	-0.01	-0.01	0.06	-0.01	-0.00	-0.01
	0.00	-0.01	-0.01	-0.01	-0.01	-0.01	-0.00	-0.00	-0.01
	-0.00	0.08	-0.01	-0.01	-0.01	-0.02	-0.01	-0.01	0.00
h)	0.00	-0.04	-0.01	-0.01	0.00	-0.01	-0.00	-0.00	-0.01
	-0.03	0.78	0.10	-0.02	-0.02	-0.03	-0.01	-0.01	-0.02
	-0.01	0.03	0.32	-0.01	-0.01	-0.01	0.00	-0.01	0.00
	-0.01	-0.02	0.07	0.29	-0.01	-0.02	0.02	-0.01	-0.01
	-0.00	-0.02	-0.00	0.05	0.03	-0.02	0.07	-0.01	-0.01
	-0.01	-0.03	-0.02	-0.02	-0.02	0.24	-0.01	0.00	-0.02
	-0.00	-0.01	-0.01	-0.01	-0.01	0.06	0.13	-0.00	-0.01
	-0.00	-0.01	0.01	-0.01	-0.01	-0.01	-0.00	0.19	-0.01
	-0.01	0.12	0.00	0.02	-0.01	-0.02	-0.01	0.13	0.34
i)	0.00	-0.01	0.00	0.00	0.00	-0.01	-0.00	-0.00	-0.01
	-0.01	0.24	-0.02	-0.02	-0.01	-0.03	0.01	-0.01	0.19
	0.00	-0.02	0.00	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01
	0.00	-0.02	-0.01	-0.02	-0.01	-0.02	-0.01	-0.01	-0.01
	0.00	-0.01	0.00	0.02	-0.01	-0.02	0.03	-0.01	-0.01
	-0.01	-0.03	-0.02	-0.02	-0.02	0.21	-0.01	0.00	-0.02
	-0.00	-0.00	-0.01	-0.01	-0.01	0.08	0.19	-0.00	-0.01
	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	-0.00	0.42	0.15
	-0.00	-0.02	-0.01	-0.01	-0.01	-0.02	-0.01	-0.01	0.16

receive if we already know that B has occurred and are told that A has occurred is

$$I(A \mid B) = -\log P(A \mid B). \tag{10}$$

The contribution of B to the information about A is expressed by the mutual information

$$I(A; B) = I(A) - I(A \mid B) = \log \frac{P(A \mid B)}{P(A)}. \tag{11}$$

Note that if A is highly correlated with B, $P(A \mid B)$ should be close to one, so that $I(A \mid B)$ is close to zero, making $I(A; B)$ high (close

TABLE III
MUTUAL INFORMATION COEFFICIENTS DERIVED FROM FIG. 2(a)

	0.00	-0.05	0.01	0.01	0.01	-0.02	-0.03	-0.09	-0.01
	-0.03	0.43	-1.00	-1.00	-0.32	-0.65	-0.06	-1.00	-1.00
	0.01	-1.00	0.08	-1.00	0.01	-1.00	-1.00	-1.00	-1.00
	0.01	-1.00	-1.00	-1.00	0.17	-1.00	-1.00	-1.00	-1.00
a)	0.01	-0.35	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	-0.02	-0.61	-0.21	-0.29	-1.00	0.40	0.36	-1.00	-1.00
	-0.01	0.10	-1.00	-1.00	0.30	-1.00	0.67	-1.00	-1.00
	-0.05	-1.00	-1.00	-1.00	-1.00	0.07	-1.00	0.98	-1.00
	-0.08	0.47	-1.00	-1.00	-1.00	-1.00	-1.00	0.65	0.53
	0.00	-0.16	-0.04	-0.03	-0.00	-0.02	-0.02	-0.01	-0.10
	-0.18	0.64	0.18	-1.00	-0.44	-0.74	-0.42	-1.00	0.39
	-0.07	0.36	0.69	0.37	-0.09	-1.00	-1.00	0.22	0.02
	-0.03	-1.00	-0.15	0.60	0.32	-1.00	-1.00	-1.00	0.15
b)	0.01	-0.71	-1.00	-1.00	0.29	-1.00	-1.00	-1.00	-0.42
	-0.03	-0.52	-0.23	-1.00	-1.00	0.44	0.34	-1.00	-1.00
	-0.02	-0.23	0.09	0.19	0.44	-0.09	0.59	-1.00	-1.00
	-0.05	-1.00	-1.00	-1.00	-1.00	0.04	-1.00	0.82	0.62
	-0.03	-1.00	0.06	-1.00	-1.00	-1.00	-1.00	-1.00	0.68
	0.00	-0.03	-0.07	-0.10	-0.06	-0.03	-0.01	0.01	-0.01
	-0.05	0.44	0.33	-1.00	-1.00	-0.49	-1.00	-1.00	0.32
	-0.02	-1.00	0.60	0.36	-0.07	-1.00	-1.00	-1.00	-1 00
	-0.10	-1.00	0.44	0.78	0.26	-1.00	-1.00	-1.00	-1.00
c)	-0.03	-0.00	-0.23	0.21	0.69	-1.00	-1.00	-1.00	-1.00
	-0.04	-0.53	-0.48	-1.00	0.39	0.46	0.32	-1.00	-1.00
	-0.00	-0.06	0.01	0.23	-0.10	0.06	-1.00	-1.00	-1.00
	0.01	-1.00	0.39	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	0.01	-1.00	0.29	0.10	-0.14	-1.00	-1.00	-1.00	0.06
	0.00	-0.04	0.01	-0.02	-0.06	-0.14	-0.02	-0.06	0.00
	-0.03	0.43	-1.00	-1.00	-0.49	-0.90	-0.09	-1.00	-1.00
	-0.01	-1.00	0.38	-1.00	0.14	-0.62	-1.00	-1.00	0.42
	-0.02	-1.00	-1.00	0.59	0.02	-1.00	-1.00	-1.00	0.18
d)	-0.04	-1.00	-1.00	-1.00	0.73	0.26	-0.39	-1.00	-0.27
	-0.13	-0.73	-0.46	-1.00	0.38	0.66	-1.00	-1.00	-1.00
	-0.06	-1.00	-1.00	0.25	-1.00	0.41	0.79	-1.00	-1.00
	0.00	-1.00	-1.00	-1.00	-1.00	0.01	-1.00	0.58	-1.00
	-0.04	0.42	-1.00	-1.00	-0.17	-1.00	-1.00	0.75	-0.19
	0.01	-0.21	-0.15	-0.13	-0.16	-0.18	-0.08	-0.14	-0.16
	-0.21	0.68	0.16	-1.00	-0.51	-0.99	-0.43	-1.00	-1.00
	-0.15	0.16	0.91	0.26	-0.08	-0.71	-1.00	-1.00	-1.00
	-0.13	-1.00	0.26	0.83	0.01	-1.00	-1.00	-1.00	-1.00
e)	-0.16	-0.51	-0.08	0.01	0.93	0.22	-0.40	-1.00	-0.23
	-0.18	-0.99	-0.71	-1.00	0.22	0.70	-1.00	-1.00	-1.00
	-0.08	-0.43	-1.00	-1.00	-0.40	-1.00	0.99	-1.00	-1.00
	-0.14	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	1.00	-1.00
	-0.16	-1.00	-1.00	-1.00	-0.23	-1.00	-1.00	-1.00	0.88
	0.00	-0.03	-0.01	-0.02	-0.04	-0.14	-0.06	0.00	-0.04
	-0.04	0.43	-1.00	-1.00	-1.00	-0.73	-1.00	-1.00	0.42
	0.01	-1.00	0.38	-1.00	-1.00	-0.46	-1.00	-1.00	-1.00
	-0.02	-1.00	-1.00	0.59	-1.00	-1.00	0.25	-1.00	-1.00
f)	-0.06	-0.49	0.14	0.02	0.73	0.38	-1.00	-1.00	-0.17
	-0.15	-0.90	-0.62	-1.00	0.26	0.66	0.41	0.01	-1.00
	-0.02	-0.09	-1.00	-1.00	-0.39	-1.00	0.79	-1.00	-1.00
	-0.06	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	0.58	0.75
	0.00	-1.00	0.42	0.18	-0.27	-1.00	-1.00	-1.00	-0.19
	0.00	-0.05	-0.02	-0.10	-0.03	-0.04	-0.00	0.01	0.01
	-0.03	0.43	-1.00	-1.00	-0.00	-0.53	-0.06	-1.00	-1.00
	-0.07	0.33	0.60	0.44	-0.23	-0.48	0.01	0.39	0.29
	-0.10	-1.00	0.36	0.78	0.21	-1.00	0.23	-1.00	0.10
g)	-0.06	-1.00	-0.07	0.26	0.69	0.39	-0.10	-1.00	-0.14
	-0.03	-0.49	-1.00	-1.00	-1.00	0.45	0.06	-1.00	-1.00
	-0.01	-1.00	-1.00	-1.00	-1.00	0.32	-1.00	-1.00	-1.00
	0.01	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	-0.01	0.32	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	0.06
	0.00	-0.18	-0.07	-0.03	0.01	-0.03	-0.02	-0.05	-0.03
	-0.16	0.64	0.36	-1.00	-0.71	-0.52	-0.23	-1.00	-1.00
	-0.04	0.18	0.69	-0.15	-1.00	-0.23	0.09	-1.00	0.06
	-0.03	-1.00	0.37	0.60	-1.00	-1.00	0.19	-1.00	-1.00
h)	-0.00	-0.44	-0.09	0.32	0.29	-1.00	0.44	-1.00	-1.00
	-0.02	-0.74	-1.00	-1.00	-1.00	0.44	-0.09	0.04	-1.00
	-0.02	-0.42	-1.00	-1.00	-1.00	0.34	0.59	-1.00	-1.00
	-0.01	-1.00	0.22	-1.00	-1.00	-1.00	-1.00	0.82	-1.00
	-0.10	0.39	0.02	0.15	-0.42	-1.00	-1.00	0.62	0.68
	0.00	-0.03	0.01	0.01	0.01	-0.02	-0.01	-0.05	-0.08
	-0.05	0.42	-1.00	-1.00	-0.35	-0.61	0.10	-1.00	0.47
	0.01	-1.00	0.08	-1.00	-1.00	-0.21	-1.00	-1.00	-1.00
	0.01	-1.00	-1.00	-1.00	-1.00	-0.29	-1.00	-1.00	-1.00
i)	0.01	-0.32	0.01	0.17	-1.00	-1.00	0.30	-1.00	-1.00
	-0.02	-0.65	-1.00	-1.00	-1.00	0.41	-1.00	0.07	-1.00
	-0.03	-0.06	-1.00	-1.00	-1.00	0.36	0.67	-1.00	-1.00
	-0.09	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	0.98	0.65
	-0.01	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	0.53

to $I(A)$); while if A is negatively correlated with B, $P(A \mid B)$ will be close to zero, so that $I(A; B)$ will be very small.

An estimate of $I(A; B)$ for the labels in the relaxation process can be derived from (11) using (6) and (8):

$$I_{ij}(\lambda; \lambda') = \log \frac{n \sum_{(x,y)} P_{(x,y)}(\lambda) P_{(x+i,y+j)}(\lambda')}{\sum_{(x,y)} P_{(x,y)}(\lambda) \sum_{(x,y)} P_{(x,y)}(\lambda')}. \tag{12}$$

The I_{ij} can be used as estimates for the P_{ij} if we can insure that the

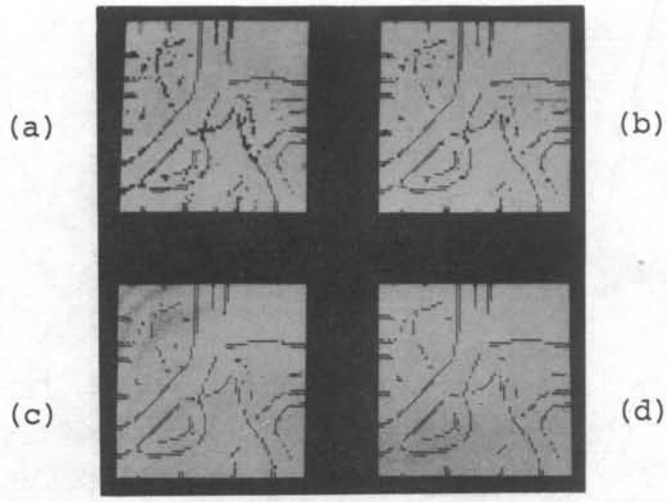

Fig. 5. Analogous to Fig. 2, but using mutual information coefficients.

updated probabilities of (3) will not be negative.[2] Extreme values of $I_{ij}(\lambda; \lambda')$ result only when one of the events involved is extremely rare. We shall assume here that events that cause $P(A \mid B)/P(A)$ to be outside the range $[e^{-5}, e^{5}]$ can be ignored, since either $P(A \mid B)$ is less than e^{-5} or $P(A)$ is less than e^{-5}. Thus values of $\log [P(A \mid B)/P(A)]$ can be considered to lie in the range $[-5, 5]$, and we can divide them by five to obtain coefficients in the range $[-1, 1]$.

Table III shows the coefficients derived in this way from Fig. 2(a), and Fig. 5 shows iterations 0 (= same as Fig. 2(a)), 2, 6, and 12 of the relaxation process using these coefficients. The results are quite good; few noise points have survived, but the curves are generally well preserved.

The similar results obtained using weighted correlation and mutual information can be explained by noting that formulas (5) and (12) are related to one another, at least approximately. In fact, from (5) we can derive

$$R^*_{ij}(\lambda, \lambda') = \frac{[1 - \bar{P}(\lambda)][1 - \bar{P}(\lambda')]}{\frac{1}{n}\sigma(\lambda)\frac{1}{n}\sigma(\lambda')} \cdot \left[\frac{1}{n}\sum_{(x,y)} P_{(x,y)}(\lambda) P_{(x+i,y+j)}(\lambda') - \bar{P}(\lambda')\right]. \tag{13}$$

On the other hand, by using a first-order approximation to $\log x$, namely $x - 1$ (using the series $\log x = (x-1) - \frac{1}{2}(x-1)^2 \cdots$), we can derive from (12)

$$I_{i,j}(\lambda; \lambda') = \frac{\frac{1}{n}\sum_{(x,y)} P_{(x,y)}(\lambda) P_{(x+i,y+j)}(\lambda')}{\bar{P}(\lambda)\bar{P}(\lambda')} - 1$$
$$= \frac{1}{\bar{P}(\lambda)\bar{P}(\lambda')}\left[\frac{1}{n}\sum_{(x,y)} P_{(x,y)} P_{(x+i,y+j)}(\lambda') - \bar{P}(\lambda)\bar{P}(\lambda')\right]. \tag{14}$$

In both (13) and (14) the expression

$$\left[\frac{1}{n}\sum_{(x,y)} P_{(x,y)}(\lambda) P_{(x+i,y+j)}(\lambda') - \bar{P}(\lambda)\bar{P}(\lambda')\right]$$

[2] The discussion of the relaxation process by Hummel [4] views the expression $[1 + q_i^k(\lambda)]$ in (3) as a first approximation to $\exp [q_i^k(\lambda)]$. If we used exp, the problem of negative values in (3) would not exist.

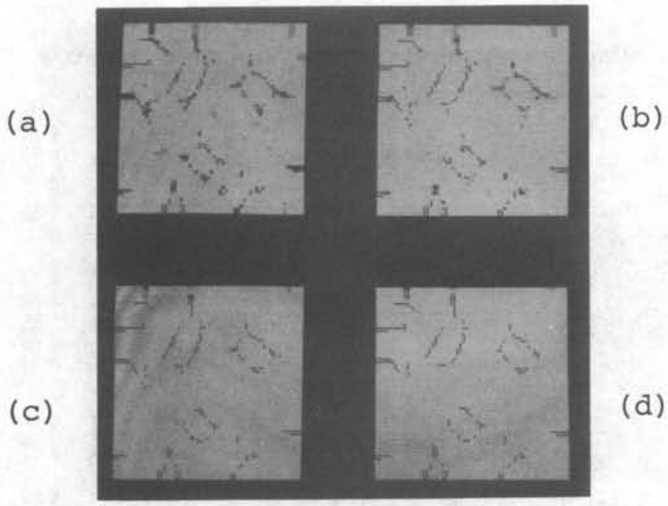

Fig. 6. Analogous to Fig. 5, for the image in Fig. 1(d).

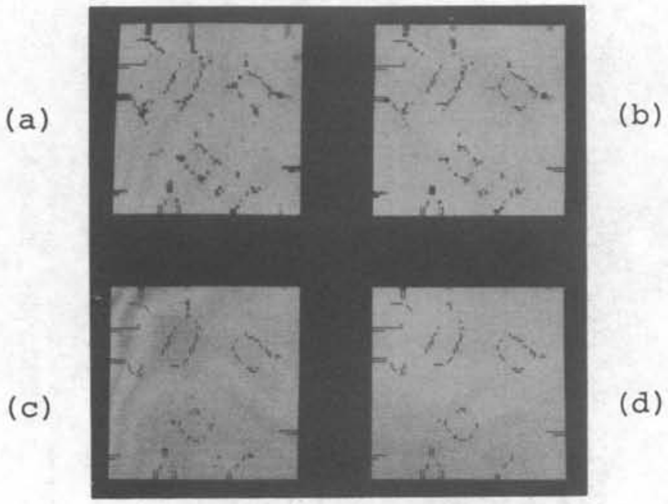

Fig. 7. Same as Fig. 6, but using coefficients derived from Fig. 5(a) rather than from Fig. 6(a).

appears, multiplied by positive expressions that depend similarly on $P(\lambda)$ and $P(\lambda')$.

Mutual information is, of course, not the only possible choice; many other functions meet the criteria for compatibility functions. However, mutual information is certainly a very simple choice, since it is simply the log of the ratio of conditional frequency to unconditional frequency.

VI. The Generality of the Coefficients

The methods described in Sections IV and V indicate that useful compatibility coefficients for relaxation curve enhancement of a given picture can be derived by analyzing the initial curve probabilities for that picture. We shall now demonstrate that these coefficients can also be used to perform the same relaxation process on other pictures. The experiments described in this section use the mutual information coefficients defined in Section V.

Initial line and "no line" probabilities were derived from the image in Fig. 1(d) by applying nonlinear line detectors; the highest probability at each point is displayed symbolically in Fig. 6(a), which is analogous to Figs. 2(a)–5(a). The coefficients derived from Fig. 6(a) using (12) are shown in Table IV. Fig. 6(b)–(d) shows iterations 2, 6, and 12 of the relaxation process using these coefficients, applied to Fig. 6(a). The results are analogous to those in Fig. 5; ambiguity has been reduced, and the basic region outlines have been preserved.

Fig. 7 shows the results of applying the coefficients of Table III (derived from Fig. 5(a)) to Fig. 6(a), and Fig. 8 shows the results of applying the Table IV coefficients to Fig. 5(a). These results are very similar to those of Figs. 5 and 6. Relaxation curve enhancement seems to work about as well when the coefficients are derived from a picture of an entirely different type, as long as it contains a reasonable set of curves.

On the other hand, one cannot indiscriminately use relaxation coefficients derived from one line detection operator to perform curve enhancement on the output of another operator. Fig. 9

TABLE IV
Mutual Information Coefficients Derived from Fig. 6(a)

	0.00	-0.03	0.00	0.01	-0.04	-0.02	-0.07	-0.08	-0.05
	-0.01	0.52	-1.00	-1.00	0.46	-1.00	0.31	0.36	0.02
	-0.03	0.25	0.37	-1.00	0.67	0.29	-1.00	-1.00	0.45
a)	0.01	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	0.01	-1.00	-1.00	-1.00	0.09	-1.00	-1.00	-1.00	-1.00
	-0.03	-1.00	-1.00	-1.00	-1.00	0.53	0.46	-1.00	0.20
	-0.03	-1.00	-1.00	-1.00	0.39	-1.00	0.80	0.57	-1.00
	-0.05	0.36	-1.00	-1.00	-1.00	-1.00	0.66	0.84	-1.00
	-0.07	-1.00	-1.00	-1.00	-1.00	0.24	-1.00	0.84	0.87
	0.00	-0.10	-0.07	-0.03	-0.03	-0.03	-0.02	-0.01	-0.07
	-0.08	0.73	0.46	-1.00	-1.00	-0.09	-1.00	-1.00	0.47
	-0.10	0.52	0.83	0.65	0.66	0.01	-1.00	-1.00	-1.00
b)	-0.02	-1.00	-1.00	0.83	0.65	-1.00	-1.00	-1.00	-1.00
	0.01	-1.00	-1.00	-1.00	0.11	-1.00	-1.00	-1.00	-1.00
	-0.04	-0.12	-1.00	-1.00	-1.00	0.56	0.42	0.14	-1.00
	-0.01	-1.00	-1.00	-1.00	0.52	-1.00	0.62	-1.00	-1.00
	-0.02	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	0.60	0.75
	-0.04	-1.00	0.36	-1.00	-1.00	0.37	-1.00	-1.00	0.83
	0.00	-0.03	-0.09	-0.06	-0.08	-0.04	-0.01	0.00	-0.01
	-0.03	0.56	0.66	-1.00	-1.00	-0.08	0.15	-1.00	0.47
	-0.02	-1.00	0.66	0.67	-1.00	-1.00	-1.00	-1.00	-1.00
	-0.02	-0.61	-1.00	0.92	-1.00	-1.00	-1.00	-1.00	-1.00
c)	-0.01	-1.00	-1.00	0.76	-0.04	-1.00	-1.00	-1.00	-1.00
	-0.03	-1.00	-1.00	-1.00	0.62	0.53	0.41	0.14	-1.00
	-0.01	0.28	0.33	-1.00	0.51	-1.00	-1.00	-1.00	-1.00
	0.01	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	-0.02	-1.00	0.57	-1.00	0.76	0.09	-1.00	-1.00	-1.00
	0.00	-0.02	-0.01	0.01	-0.08	-0.11	-0.06	-0.04	-0.03
	-0.02	0.61	-1.00	-1.00	0.34	-0.10	0.37	-1.00	-1.00
	-0.02	-1.00	0.59	-1.00	0.52	-1.00	-1.00	-1.00	0.63
	-0.02	-1.00	-1.00	-1.00	0.90	0.26	-1.00	-1.00	-1.00
d)	-0.03	-1.00	-1.00	-1.00	0.64	0.54	-1.00	-1.00	-1.00
	-0.11	-1.00	-1.00	-1.00	0.44	0.75	0.34	-1.00	0.19
	-0.09	0.15	-1.00	-1.00	-1.00	0.56	0.83	0.42	0.42
	-0.01	-1.00	-1.00	-1.00	-1.00	0.35	-1.00	-1.00	-1.00
	-0.06	0.42	-1.00	-1.00	-1.00	-1.00	-1.00	0.88	0.64
	0.00	-0.17	-0.20	-0.14	-0.13	-0.18	-0.15	-0.14	-0.15
	-0.17	0.85	0.40	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	-0.20	0.40	1.00	0.35	-1.00	-1.00	-1.00	-1.00	-1.00
e)	-0.14	-1.00	0.35	1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	-0.13	-1.00	-1.00	-1.00	1.00	0.49	-1.00	-1.00	-1.00
	-0.18	-1.00	-1.00	-1.00	0.49	0.83	-1.00	-1.00	-1.00
	-0.15	-1.00	-1.00	-1.00	-1.00	-1.00	1.00	0.51	-1.00
	-0.14	-1.00	-1.00	-1.00	-1.00	-1.00	0.51	1.00	-1.00
	-0.15	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	1.00
	0.00	-0.02	-0.02	-0.02	-0.03	-0.12	-0.09	-0.01	-0.06
	-0.02	0.61	-1.00	-1.00	-1.00	-1.00	0.15	-1.00	0.42
	-0.01	-1.00	0.59	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	0.01	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
f)	-0.08	0.34	0.52	0.90	0.64	0.44	-1.00	-1.00	-1.00
	-0.12	-0.10	-1.00	0.26	0.54	0.74	0.56	0.35	-1.00
	-0.06	0.37	-1.00	-1.00	-1.00	0.34	0.83	-1.00	-1.00
	-0.04	-1.00	-1.00	-1.00	-1.00	-1.00	0.42	-1.00	0.88
	-0.03	-1.00	0.63	-1.00	-1.00	0.19	0.42	-1.00	0.64
	0.00	-0.02	-0.02	-0.02	-0.01	-0.03	-0.01	0.01	-0.02
	-0.02	0.61	-1.00	-0.61	-1.00	-1.00	0.28	-1.00	-1.00
	-0.09	0.66	0.66	-1.00	-1.00	-1.00	0.33	-1.00	0.57
	-0.06	-1.00	0.67	0.92	0.76	-1.00	-1.00	-1.00	-1.00
g)	-0.08	-1.00	-1.00	-1.00	-0.04	0.62	0.51	-1.00	0.76
	-0.03	-0.08	-1.00	-1.00	-1.00	0.55	-1.00	-1.00	0.09
	-0.01	0.15	-1.00	-1.00	-1.00	0.41	-1.00	-1.00	-1.00
	0.00	-1.00	-1.00	-1.00	-1.00	0.14	-1.00	-1.00	-1.00
	-0.01	0.47	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	0.00	-0.07	-0.10	-0.02	0.01	-0.04	-0.01	-0.02	-0.04
	-0.10	0.78	0.52	-1.00	-1.00	-0.12	-1.00	-1.00	-1.00
	-0.07	0.46	0.83	-1.00	-1.00	-1.00	-1.00	-1.00	0.36
	-0.03	-1.00	0.65	0.83	-1.00	-1.00	-1.00	-1.00	-1.00
h)	-0.03	-1.00	0.66	0.65	0.11	-1.00	0.52	-1.00	-1.00
	-0.03	-0.09	0.01	-1.00	-1.00	0.56	-1.00	-1.00	0.37
	-0.02	-1.00	-1.00	-1.00	-1.00	0.42	0.62	-1.00	-1.00
	-0.01	-1.00	-1.00	-1.00	-1.00	0.14	-1.00	0.60	-1.00
	-0.07	0.47	-1.00	-1.00	-1.00	-1.00	-1.00	0.75	0.83
	0.00	0.00	-0.03	0.01	0.01	-0.04	-0.03	-0.05	-0.07
	-0.01	0.58	0.25	-1.00	-1.00	-1.00	-1.00	0.36	-1.00
	0.00	-1.00	0.37	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	0.01	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
i)	-0.04	0.46	0.67	-1.00	0.09	-1.00	0.39	-1.00	-1.00
	-0.02	-1.00	0.29	-1.00	-1.00	0.51	-1.00	-1.00	0.24
	-0.07	0.31	-1.00	-1.00	-1.00	0.46	0.80	0.66	-1.00
	-0.08	0.36	-1.00	-1.00	-1.00	-1.00	0.57	0.84	0.84
	-0.05	0.02	0.45	-1.00	-1.00	0.20	-1.00	-1.00	0.87

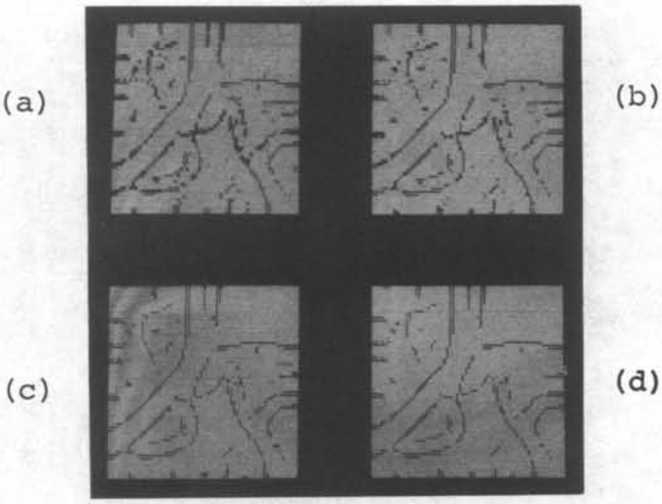

Fig. 8. Same as Fig. 5, but using coefficients derived from Fig. 6(a) rather than from Fig. 5(a).

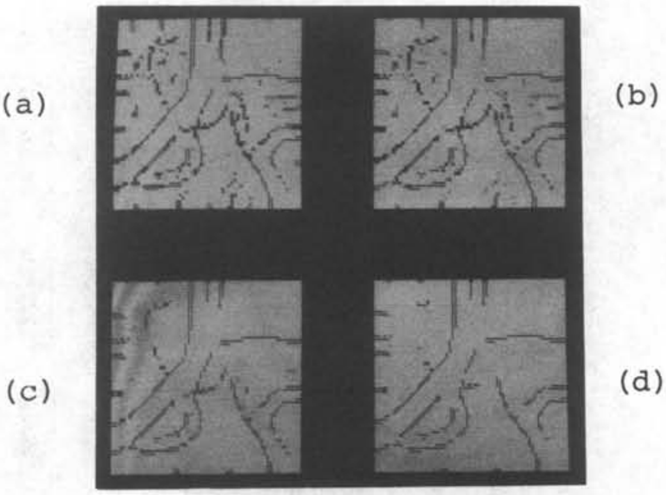

Fig. 9. Analogous to Fig. 8, but using coefficients derived from linear detector outputs rather than from nonlinear detector outputs used in previous examples.

TABLE V
MUTUAL INFORMATION COEFFICIENTS DERIVED FROM OUTPUTS OF LINEAR LINE DETECTORS APPLIED TO FIG. 1(c)

	0.01	-0.06	-0.02	-0.02	-0.04	-0.02	-0.03	-0.04	-0.05
	-0.03	0.24	0.06	0.09	-0.06	-0.08	-0.04	0.19	0.16
	-0.05	0.24	0.19	0.16	0.07	-0.02	-0.10	0.07	0.23
a)	-0.02	0.07	0.10	0.16	0.08	0.01	-0.09	-0.00	0.12
	-0.02	-0.07	-0.04	-0.03	0.19	0.09	0.14	-0.05	0.00
	-0.04	-0.09	-0.07	-0.02	0.27	0.24	0.23	-0.06	-0.05
	-0.04	-0.03	-0.00	0.04	0.24	0.17	0.20	-0.01	-0.01
	-0.03	0.16	0.06	0.02	-0.08	-0.02	0.04	0.22	0.12
	-0.05	0.27	0.10	-0.02	-0.12	-0.06	0.04	0.24	0.25
	0.01	-0.08	-0.06	-0.04	-0.06	-0.03	-0.03	-0.02	-0.08
	-0.08	0.34	0.23	0.10	-0.06	-0.09	-0.05	0.15	0.30
	-0.07	0.23	0.29	0.23	0.06	-0.02	-0.08	-0.02	0.20
b)	-0.04	0.12	0.17	0.25	0.11	0.02	-0.09	-0.02	0.09
	-0.03	-0.05	-0.01	0.05	0.24	0.14	0.13	-0.11	-0.01
	-0.05	-0.08	-0.05	-0.01	0.29	0.26	0.24	-0.07	-0.06
	-0.04	-0.04	0.01	0.03	0.24	0.18	0.22	-0.02	-0.03
	-0.03	0.14	0.06	-0.04	-0.09	-0.05	0.05	0.25	0.20
	-0.06	0.25	0.21	0.04	-0.06	-0.07	-0.02	0.15	0.30
	0.01	-0.03	-0.07	-0.05	-0.06	-0.03	-0.04	-0.01	-0.05
	-0.06	0.23	0.28	0.14	-0.04	-0.10	-0.06	0.08	0.26
	-0.04	0.13	0.24	0.23	0.02	-0.02	-0.02	-0.01	0.10
c)	-0.05	0.18	0.20	0.23	0.10	0.03	-0.01	0.04	0.09
	-0.04	-0.03	0.02	0.11	0.26	0.17	0.15	-0.10	-0.02
	-0.06	-0.07	-0.04	0.01	0.30	0.27	0.25	-0.09	-0.06
	-0.03	-0.07	-0.01	-0.02	0.22	0.17	0.23	-0.04	-0.07
	-0.03	0.10	0.10	-0.01	-0.04	-0.07	0.03	0.17	0.20
	-0.03	0.09	0.23	0.14	0.00	-0.08	-0.10	0.02	0.20
	0.01	-0.06	-0.03	-0.04	-0.06	-0.07	-0.04	-0.03	-0.04
	-0.03	0.24	0.10	0.14	-0.04	-0.08	-0.06	0.15	0.13
	-0.07	0.27	0.23	0.20	0.06	-0.03	-0.06	0.08	0.24
d)	-0.04	0.11	0.18	0.22	0.12	0.02	-0.10	-0.02	0.15
	-0.06	-0.06	-0.01	0.05	0.30	0.26	0.20	-0.07	0.02
	-0.07	-0.10	-0.04	0.01	0.28	0.34	0.24	-0.08	-0.07
	-0.06	-0.05	-0.01	0.03	0.23	0.27	0.29	0.02	-0.07
	-0.02	0.12	0.01	0.02	-0.13	-0.06	0.00	0.21	0.07
	-0.05	0.27	0.09	0.01	-0.10	-0.07	-0.03	0.23	0.24
	0.01	-0.09	-0.07	-0.05	-0.07	-0.07	-0.05	-0.03	-0.07
	-0.09	0.35	0.24	0.11	-0.05	-0.09	-0.05	0.15	0.28
	-0.07	0.24	0.32	0.22	0.05	-0.04	-0.04	-0.00	0.20
e)	-0.05	0.11	0.22	0.30	0.11	0.02	-0.04	-0.06	0.06
	-0.07	-0.05	0.05	0.11	0.33	0.28	0.21	-0.16	-0.04
	-0.07	-0.09	-0.04	0.02	0.28	0.36	0.27	-0.08	-0.08
	-0.05	-0.05	-0.04	-0.04	0.21	0.27	0.32	0.04	-0.05
	-0.03	0.15	-0.00	-0.06	-0.16	-0.08	0.04	0.30	0.19
	-0.07	0.28	0.20	0.06	-0.04	-0.08	-0.05	0.19	0.33
	0.01	-0.03	-0.07	-0.04	-0.06	-0.07	-0.06	-0.02	-0.05
	-0.06	0.24	0.27	0.11	-0.06	-0.10	-0.05	0.12	0.27
	-0.03	0.10	0.23	0.18	-0.01	-0.04	-0.01	0.01	0.09
f)	-0.04	0.14	0.20	0.22	0.05	0.01	0.03	0.02	0.01
	-0.07	-0.04	0.06	0.12	0.31	0.28	0.23	-0.13	-0.10
	-0.07	-0.08	-0.03	0.01	0.26	0.35	0.28	-0.06	-0.07
	-0.04	-0.06	-0.06	-0.10	0.20	0.25	0.29	-0.00	-0.03
	-0.03	0.15	0.08	-0.02	-0.07	-0.07	0.02	0.21	0.23
	-0.04	0.13	0.24	0.15	0.02	-0.07	-0.07	0.07	0.24
	0.01	-0.06	-0.04	-0.05	-0.04	-0.06	-0.03	-0.03	-0.03
	-0.03	0.23	0.13	0.17	-0.03	-0.07	-0.07	0.10	0.09
	-0.07	0.28	0.24	0.20	0.02	-0.04	-0.01	0.10	0.22
g)	-0.05	0.14	0.23	0.23	0.11	0.01	-0.02	-0.01	0.14
	-0.06	-0.04	0.02	0.10	0.26	0.30	0.22	-0.05	0.00
	-0.03	-0.10	-0.02	0.03	0.17	0.27	0.17	-0.07	-0.08
	-0.03	-0.06	-0.02	-0.01	0.15	0.25	0.23	0.03	-0.10
	-0.01	0.08	-0.02	0.03	-0.09	-0.09	-0.04	0.17	0.02
	-0.05	0.26	0.10	0.09	-0.02	-0.06	-0.07	0.20	0.20
	0.01	-0.08	-0.06	-0.04	-0.03	-0.05	-0.04	-0.03	-0.06
	-0.08	0.34	0.23	0.12	-0.05	-0.08	-0.05	0.14	0.25
	-0.06	0.23	0.29	0.17	-0.01	-0.05	0.01	0.06	0.21
h)	-0.04	0.10	0.23	0.25	0.05	-0.01	0.03	-0.04	0.04
	-0.06	-0.06	0.06	0.11	0.24	0.29	0.24	-0.09	-0.06
	-0.03	-0.09	-0.02	0.02	0.14	0.26	0.18	-0.05	-0.07
	-0.03	-0.05	-0.08	-0.09	0.13	0.24	0.22	0.05	-0.02
	-0.02	0.15	-0.02	-0.02	-0.11	-0.07	-0.02	0.25	0.15
	-0.08	0.30	0.20	0.09	-0.01	-0.06	-0.03	0.20	0.30
	0.01	-0.03	-0.05	-0.02	-0.02	-0.04	-0.04	-0.03	-0.05
	-0.06	0.24	0.24	0.07	-0.07	-0.09	-0.03	0.16	0.27
	-0.02	0.06	0.19	0.10	-0.04	-0.07	-0.00	0.06	0.10
i)	-0.02	0.08	0.16	0.16	-0.03	-0.02	0.04	0.02	-0.02
	-0.05	-0.06	0.07	0.07	0.19	0.27	0.24	-0.08	-0.12
	-0.02	-0.08	-0.02	0.01	0.09	0.24	0.17	-0.02	-0.06
	-0.03	-0.04	-0.10	-0.09	0.14	0.23	0.20	0.04	0.04
	-0.04	0.18	0.07	-0.00	-0.05	-0.05	-0.02	0.22	0.23
	-0.05	0.16	0.23	0.12	0.00	-0.05	-0.01	0.12	0.25

shows what happens when the coefficients derived from the outputs of linear line detectors (see Section III) are applied to the initial probabilities derived from nonlinear detector outputs. Ambiguity is reduced, but many of the weaker curves are destroyed. (The coefficients themselves are shown in Table V.)

A quantitative or comparative evaluation of the results obtained using the various sets of coefficients has not been attempted here. We could have, for example, compared the results obtained using mutual information coefficients with those obtained using a simple continuation measure (e.g., based on a product of cosines) as in [3]. However, note that the continuation coefficients could equally well have been used with any line detection operator, whereas the information coefficients are specific to a particular operator, so that it seems unfair to compare the two methods. We could also have evaluated the reduction in ambiguity produced by the relaxation process by using, e.g., an entropy measure; but note that this measure is very low when there are (e.g.) only "no line" responses (as happens in the case of Fig. 2), so that it might rate the unweighted correlation coefficients as very successful. The problem of quantitatively evaluating the results of relaxation processes still lacks a satisfactory solution.

VII. CONCLUDING REMARKS

The results reported here indicate that usable compatibility coefficients for some types of relaxation processes can be derived by statistical analysis of the initial label probabilities. This process appears to be robust, in the sense that the coefficients can be computed in at least two different ways (weighted correlation coefficients or mutual information values) and that coefficients computed from one image will also give good performance on other images. Thus it does not seem to be necessary to derive the coefficients by analyzing a large ensemble of images. It would be of interest to conduct further studies of this approach in connection with other applications of relaxation processes [2].

ACKNOWLEDGMENT

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Iterative Histogram Modification, 2

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***Abstract*—Histogram peaks can be sharpened using an iterative process in which large bins grow at the expense of nearby smaller bins. The modified histogram will consist of a few spikes corresponding to the peaks of the original histogram. The image corresponding to the modified histogram is often almost undistinguishable from the original image. The small number of different gray levels in that image can be used to facilitate approximating or segmenting it.**

I. INTRODUCTION

The histogram of an image is the discrete distribution function of the gray levels of the pixels in it. This correspondence describes a process for sharpening peaks on an image's histogram. It supplements preliminary work by Rosenfeld and Davis [1]. The process thins each peak on the original histogram into a spike. The image, corresponding to the modified histogram, has only a few gray levels. These gray levels correspond to the spikes in the modified histogram. The process can also generate a spike from the "shoulder" of a peak. Such shoulders are created by small peaks close to bigger ones; the process provides a cheap method of discovering such hidden peaks. The resulting image is a mapping of the original image into very few gray levels corresponding to the spikes found. This mapping provides an initial segmentation of the image, each segment corresponding to a spike in the histogram. Even though the modified image generally consists of very few gray levels, no deterioration in the image detail is seen. This should make possible efficient coding of the image without noticeable deterioration in its quality.

II. THE ALGORITHM

The algorithm described below operates on a one-dimensional histogram, but has a natural generalization to any number of dimensions. Thus it could be used to process three-dimensional color histograms or histograms based on additional pixel properties besides gray level. (This generalization was suggested by E. Riseman in a personal communication.)

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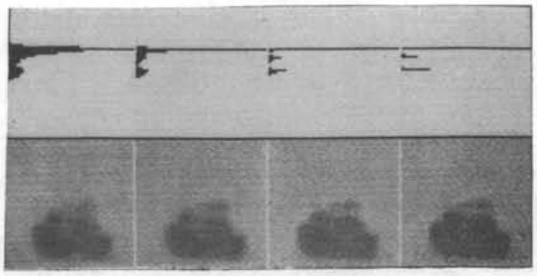

Fig. 1. Iterations 0, 1, 2, 4 of peak sharpening process.

Let B_i be the number of pixels having gray level i. For each histogram bin i, the neighboring $2r$ (an input parameter) bins $i \pm j$ on each side of i ($j = 1, 2, \cdots, r$) are examined. If B_i is greater than the average A of $B_{i+1}, \cdots, B_{i+r}$ (and similarly on the other side of i), we compute the ratio $X = (B_i - A)/B_i$, which specifies the fraction of pixels whose gray levels will be shifted towards i. Then the following gray-level changes are executed:

$$B_{i+r} \cdot X \text{ from } i + r \text{ to } i + r - 1;$$

$$B_{i+r-1} \cdot X \text{ from } i + r - 1 \text{ to } i + r - 2; \cdots;$$

$$B_{i+1} \cdot X \text{ from } i + 1 \text{ to } i.$$

The entire process is then iterated.

In order to minimize the changes in gray levels and to preserve their original order (i.e., to preserve "darker than" and "lighter than" relations), a "history" of pixel movement is kept. A matrix H is created in which element $H(\alpha, \beta)$ indicates the number of pixels currently at gray level α that had original gray level β. Initially,

$$H(i, j) = \begin{cases} 0, & i \neq j \\ \text{number of pixels with gray level } i, & i = j. \end{cases}$$

The algorithm performs gray-level changes on H only (*not* on the image). When transferring pixels from gray level α to gray level β, the pixels transferred first are those whose origin is closest to β. Finally, the image is transformed by changing $H(\alpha, \beta)$ pixels from gray level β to gray level α.

III. BANDWIDTH COMPRESSION

An immediate application of the algorithm described previously is to provide a segmentation of the image into a few gray levels. Simple images such as tanks (see Figs. 1, 6, 7) can be represented by three or four gray levels, thus reducing the number of bits per pixel from six to two. Figs. 2–5 show that even more complicated images can be represented by about eight distinct gray levels.

Efficient encoding schemes can be used to further improve compression. From the final histogram we can derive a Huffman coding [2] for the image. This coding gives about 1.4 bits per pixel for the tank images in Figs. 6 and 7 and 2.1 bits per pixel for Figs. 4 and 5. Run length coding can also be used, since the reduction in the number of gray levels favors longer runs.

IV. EXAMPLES

Fig. 1 shows the steps in the creation of spikes from the original histogram. Displayed are the original image and its histogram and the images produced after one, two, and four iterations. Figs. 2–7 each consist of an original image and, to its right, the images