Precoded Integer-Forcing Universally Achieves the MIMO Capacity to Within a Constant Gap

Or Ordentlich
Joint work with Uri Erez

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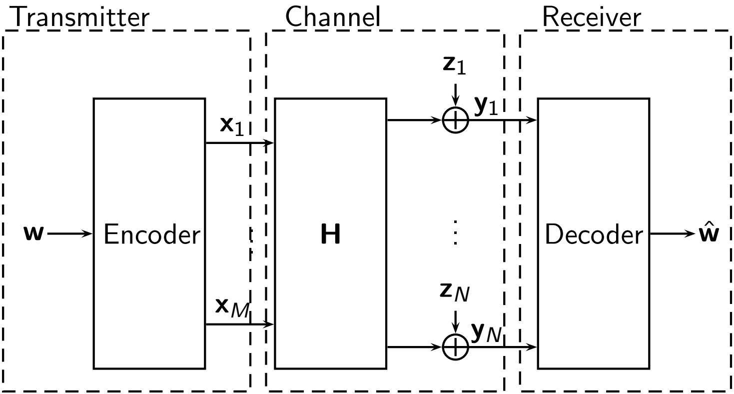
ITW, Seville, Spain

The MIMO Channel



$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$$

- $\mathbf{H} \in \mathbb{C}^{N\times M}$, $\mathbf{x} \in \mathbb{C}^{M\times 1}$ and $\mathbf{z} \sim \mathcal{CN}(0, \mathbf{I}_N)$.
- Power constraint is $\mathbb{E}\|\mathbf{x}\|^2 \leq M \cdot \text{SNR}$.

The MIMO Channel

Closed-loop

$$C = \max_{\mathbf{Q} \succ 0 \;:\; \mathrm{trace}\,\mathbf{Q} \leq M \cdot \mathsf{SNR}} \log\det\left(\mathbf{I} + \mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H}\right)$$

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Open-loop

Optimizing $\mathbf{Q}$ is impossible. Isotropic transmission $\mathbf{Q} = \mathsf{SNR} \cdot \mathbf{I}$ is a reasonable idea and gives

$$C_{\mathsf{WI}} = \log\det\left(\mathbf{I} + \mathsf{SNR}\mathbf{H}^{\dagger}\mathbf{H}\right)$$

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Definition: Compound channel

The compound MIMO channel with capacity C_{WI} consists of the set of all channel matrices

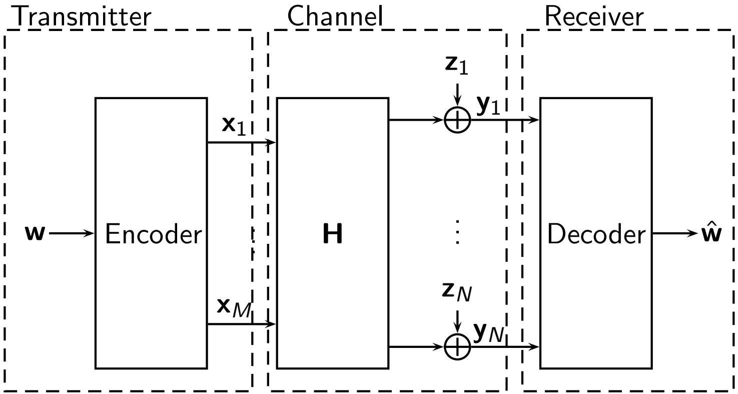
$$\mathbb{H}(C_{\mathsf{WI}}) = \left\{\mathbf{H}\in\mathbb{C}^{N\times M} \;:\; \log\det\left(\mathbf{I}+\mathsf{SNR}\mathbf{H}^{\dagger}\mathbf{H}\right)=C_{\mathsf{WI}}\right\}$$

The MIMO Channel

How can we approach the compound channel capacity in practice*?

*practice = scalar AWGN coding & decoding + linear pre/post processing

Decoupling Decoding from Equalization



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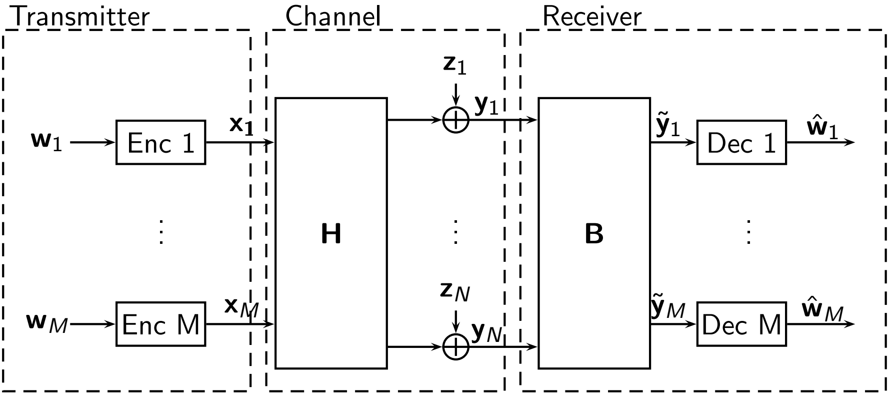

Split $\mathbf{w}$ to M messages $\mathbf{w}_1, \cdots \mathbf{w}_M$

encode each message separately

equalize channel and decode each message separately

The MIMO Channel - Practical Schemes

Closed-loop

Can transform the channel to a set of parallel SISO channels via SVD or QR

- Use standard AWGN encoders and decoders (e.g., turbo, LDPC) for the SISO channels
- Gap to capacity is the same as that of the AWGN codes

The MIMO Channel - Practical Schemes

Compound channel

Much less is known...

- Can still apply QR at the receiver, but how should the transmitter allocate rates to the different streams?
- Can also apply linear equalization (ZF or MMSE), but loss can be large

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Statistical approach

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$$\mathbb{E}_{\mathbf{H}}\left(P_e\right) = \mathbb{E}_{C_{\mathrm{WI}}}\left(\mathbb{E}_{\mathbf{H}}\left(P_e \middle| C_{\mathrm{WI}}\right)\right)$$

The MIMO Channel - DMT

Diversity-multiplexing tradeoff (Zheng-Tse IT03)

- Introduced as a physical characterization of the channel
- Has become a benchmark for assessing practical coding schemes

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Weakness #1 - (lack of) robustness to channel statistics

- DMT optimality of a scheme does not translate to performance guarantees for specific channel realizations
  $\Longrightarrow$ Can design a scheme to work well only for typical channels

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Solution: approximately universal codes

- Introduced by Tavildar and Vishwanath (IT06)
- DMT optimal regardless of the channel statistics

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As a benchmark, DMT is powerful, but has two weaknesses:

Weakness #2 - crude measure of error probability

- For "good" channel realizations, the error probability is only required to be smaller than the outage probability
  $\Longrightarrow$ A scheme with short block length (essentially "uncoded") can be DMT optimal

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When not in outage, we want communication to be reliable

Precoded Integer-Forcing

This Work

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Constant gap-to-capacity also implies

- DMT optimality
- Constant gap to the outage capacity for any channel statistics

Precoded Integer-Forcing

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A low-complexity scheme that achieves the compound MIMO **capacity** to within a constant gap

Constant gap-to-capacity also implies

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Main result

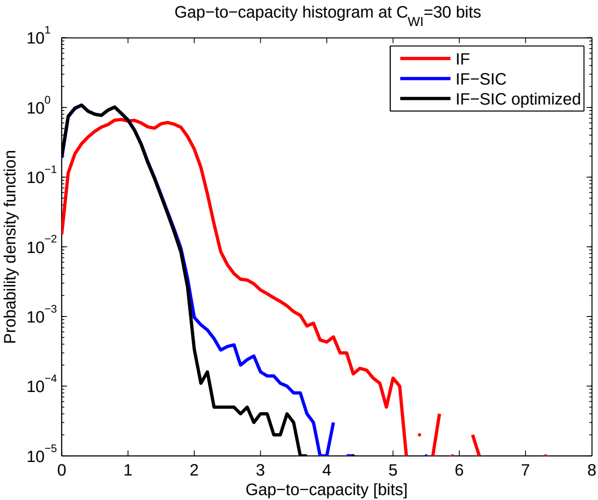
IF equalization with space-time coded transmission can achieve any rate

$$R < C_{\text{WI}} - \Gamma\left(\delta_{\min}(\mathcal{C}_{\infty}^{\text{ST}}), M\right)$$

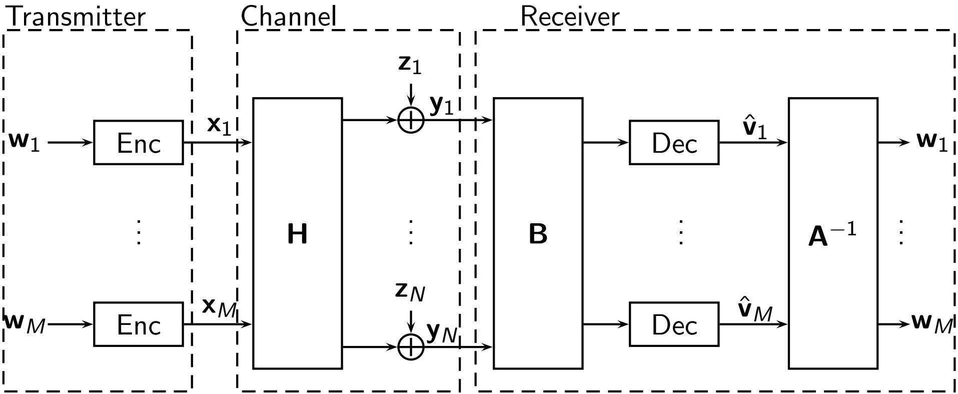
where $\Gamma\left(\delta_{\min}(\mathcal{C}_{\infty}^{\text{ST}}), M\right) \triangleq \log \frac{1}{\delta_{\min}(\mathcal{C}_{\infty}^{\text{ST}})} + 3M\log(2M^2)$

Precoded Integer-Forcing

For 2×2 Rayleigh fading with Golden code precoding

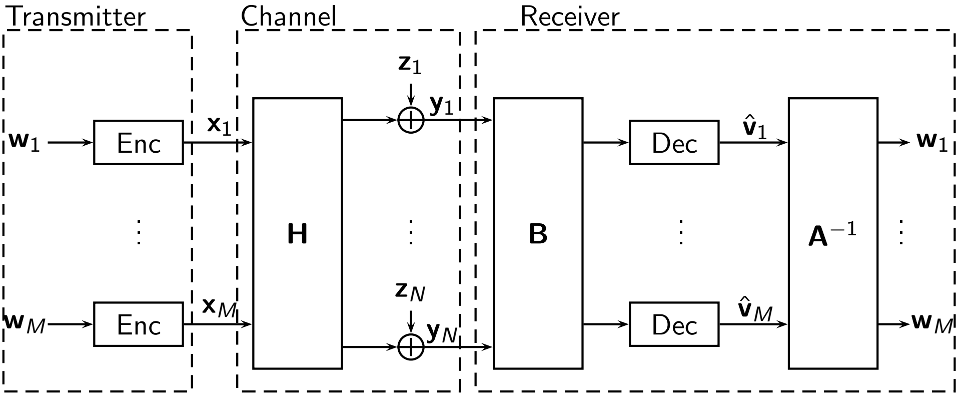


Integer-Forcing: Background



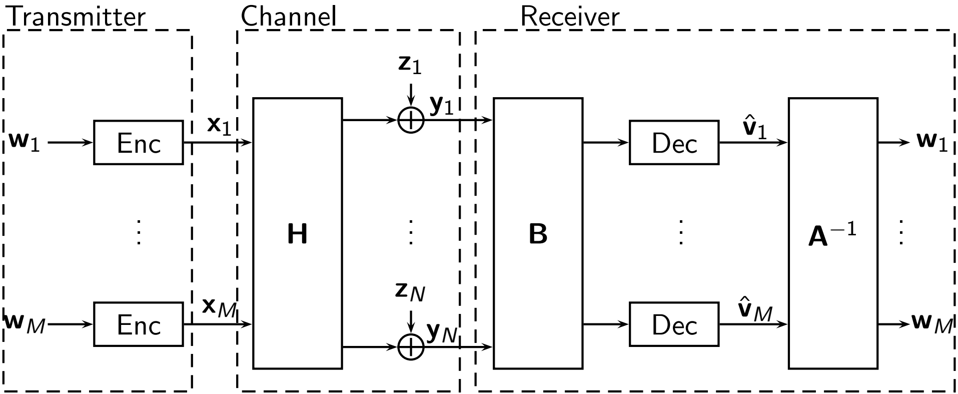
- Proposed by Zhan *et al.* ISIT2010

Integer-Forcing: Background



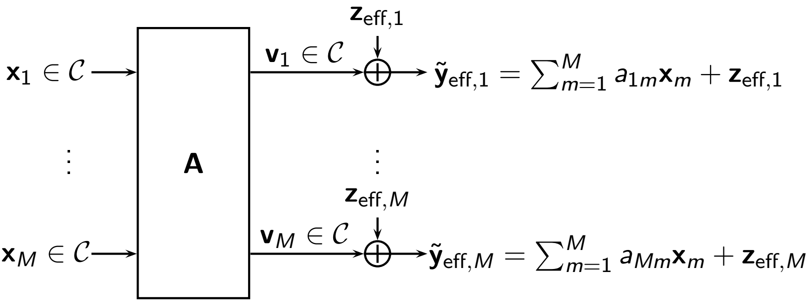
- Antennas transmit independent streams (BLAST).
- All streams are codewords from the same **same linear code** with rate R.

Integer-Forcing: Background



- Rather than equalizing $\mathbf{H}$ to identity (as in ZF or MMSE), in IF the channel is equalized to a full-rank $\mathbf{A} \in \mathbb{Z}^M + i\mathbb{Z}^M$

$$\mathbf{B} = \mathbf{A}\mathbf{H}^\dagger \left(\mathsf{SNR}^{-1}\mathbf{I} + \mathbf{H}\mathbf{H}^\dagger\right)^{-1}$$



- A linear combination of codewords with integer coefficients is a codeword
  $\Longrightarrow$ Can decode the linear combinations - remove noise
  $\Longrightarrow$ Can solve noiseless linear combinations for the transmitted streams

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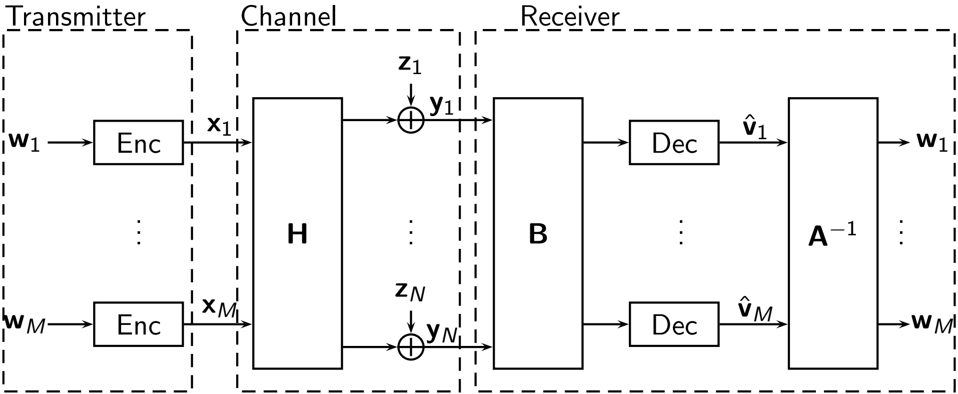

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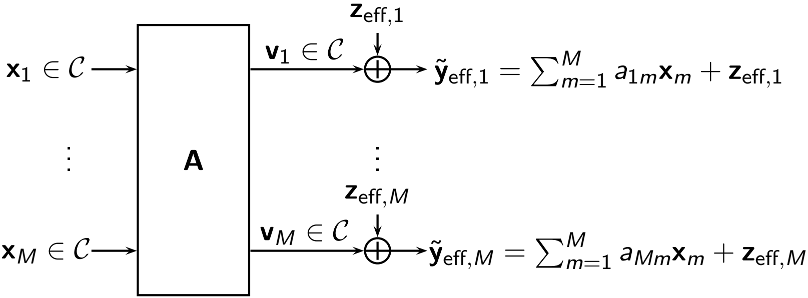

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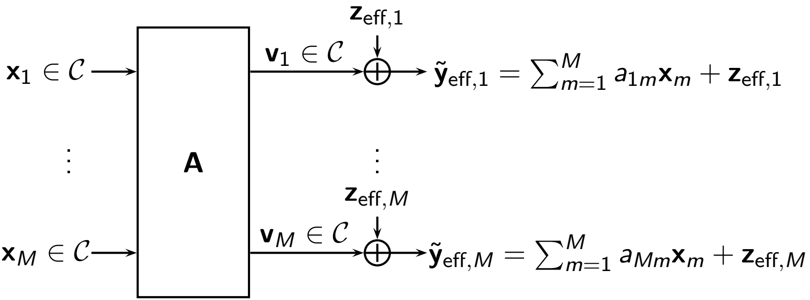
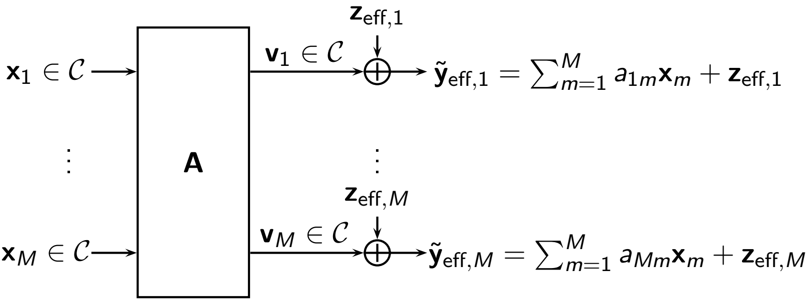


- Effective noise $\mathbf{z}_{\text{eff},k}$ has effective variance

$$\begin{aligned}\sigma^2_{\text{eff},k} &\triangleq \frac{1}{n}\mathbb{E}\left\|\mathbf{z}_{\text{eff},k}\right\|^2 \\ &= \text{SNR}\mathbf{a}_k^\dagger\left(\mathbf{I}+\text{SNR}\mathbf{H}^\dagger\mathbf{H}\right)^{-1}\mathbf{a}_k\end{aligned}$$

where $\mathbf{a}_k^\dagger$ is the kth row of $\mathbf{A}$.

Integer-Forcing: Background



- Same codebook used over all subchannels
  $\Longrightarrow$ the subchannel with the largest noise dictates the performance

$$\mathrm{SNR}_{\text{eff},k} \triangleq \frac{\mathrm{SNR}}{\sigma^2_{\text{eff},k}} = \left[\mathbf{a}_k^\dagger \left(\mathbf{I} + \mathrm{SNR}\mathbf{H}^\dagger\mathbf{H}\right)^{-1} \mathbf{a}_k\right]^{-1}$$

$$\mathrm{SNR}_{\text{eff}} \triangleq \min_{k=1,\ldots,M} \mathrm{SNR}_{\text{eff},k} = \left[\max_{k=1,\ldots,M} \mathbf{a}_k^\dagger \left(\mathbf{I} + \mathrm{SNR}\mathbf{H}^\dagger\mathbf{H}\right)^{-1} \mathbf{a}_k\right]^{-1}$$

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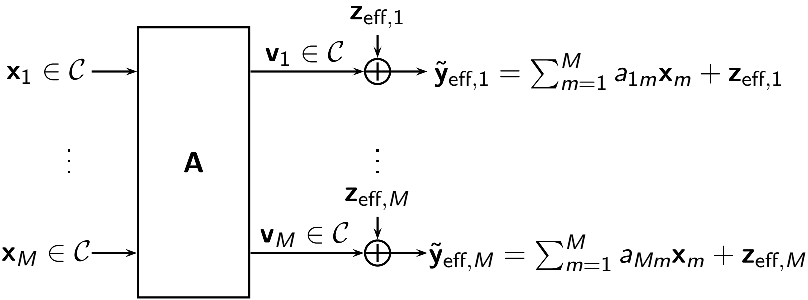
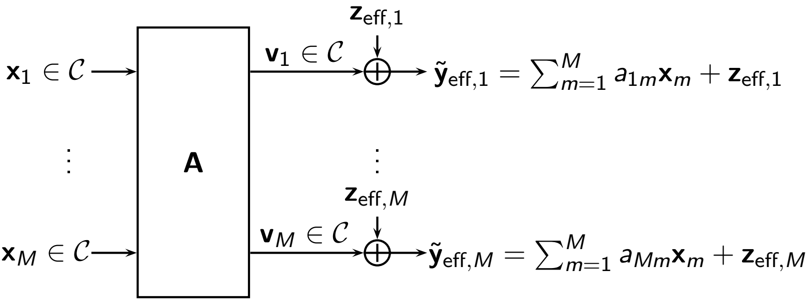


For AWGN capacity achieving nested lattice codebook $\mathcal{C}$

$$R_{\text{IF}} < M \log(\text{SNR}_{\text{eff}})$$

Integer-Forcing: Background



For AWGN capacity achieving nested lattice codebook $\mathcal{C}$

$$R_{\text{IF}} < M \log(\text{SNR}_{\text{eff}})$$

To approach C_{WI} we need $\text{SNR}_{\text{eff}} \approx 2^{\frac{C_{\text{WI}}}{M}}$

Integer-Forcing: SNR_{eff}

$$\text{SNR}_{\text{eff}} = \frac{1}{\min_{\substack{\mathbf{A}\in\mathbb{Z}^{M\times M}+i\mathbb{Z}^{M\times M} \\ \det(\mathbf{A})\neq 0}} \max_{k=1\ldots,M} \mathbf{a}_k^\dagger \left(\mathbf{I}+\text{SNR}\mathbf{H}^\dagger\mathbf{H}\right)^{-1}\mathbf{a}_k}$$

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- Does not give much insight to the dependence on $\mathbf{H}$ ☹

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- Does not give much insight to the dependence on $\mathbf{H}$ ☹
- Fortunately, using a transference theorem by Banaszczyk we can lower bound with a simple expression ☺

Integer-Forcing: SNR_{eff} via Uncoded $d_{\min}$

Theorem - SNR_{eff} bound

$$\text{SNR}_{\text{eff}} > \frac{1}{4M^2} \min_{\mathbf{a} \in \mathbb{Z}^M + i\mathbb{Z}^M \setminus \mathbf{0}} \mathbf{a}^\dagger \left(\mathbf{I} + \text{SNR}\mathbf{H}^\dagger \mathbf{H} \right) \mathbf{a}$$

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Let

$$\text{QAM}(L) \triangleq \{-L, -L+1, \ldots, L-1, L\} + i\{-L, -L+1, \ldots, L-1, L\},$$

and define $d_{\min}(\mathbf{H}, L) \triangleq \min_{\mathbf{a} \in \text{QAM}^M(L) \setminus \mathbf{0}} \|\mathbf{Ha}\|$

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Corollary

$$\mathsf{SNR}_{\text{eff}} > \frac{1}{4M^2} \min_{L=1,2,\ldots} \left(L^2 + \mathsf{SNR} d_{\min}^2(\mathbf{H}, L) \right)$$

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What can we guarantee for a specific channel realization?

Unfortunately nothing...

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Example for a bad channel

$$\mathbf{H} = \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix}$$

- $\text{SNR}_{\text{eff}} = 1$, $R_{\text{IF}} = M \log(\text{SNR}_{\text{eff}}) = 0$.
- $C_{\text{WI}} - R_{\text{IF}}$ is unbounded (as with any BLAST scheme).

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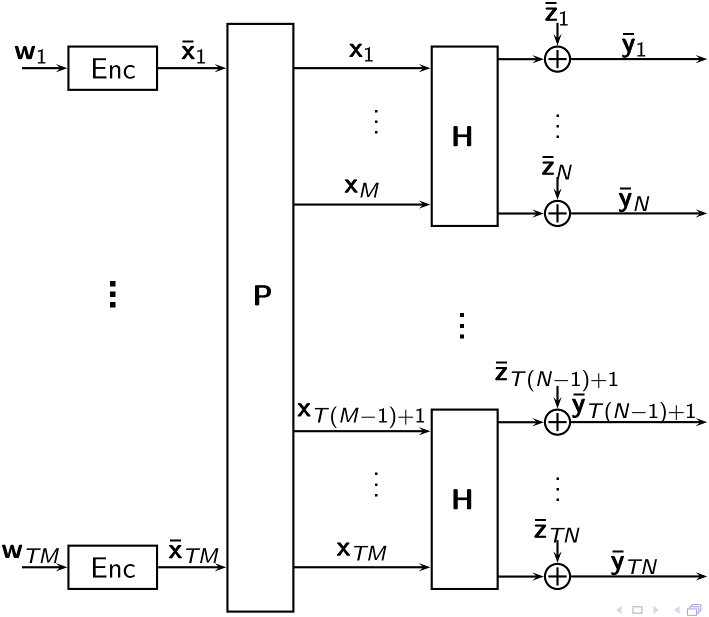
Need to precode over time for transmit diversity

Space-Time Coding/Modulation

- Instead of transmitting M independent streams of length n over n time slots, transmit MT independent streams over nT time slots
- Before transmission, precode all MT streams using a unitary matrix $\mathbf{P} \in \mathbb{C}^{MT \times MT}$.

Space-Time Coding/Modulation



Precoded Integer-Forcing

$$\bar{\mathbf{y}} = \begin{bmatrix} \mathbf{H} & 0 & \cdots & 0 \\ 0 & \mathbf{H} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H} \end{bmatrix} \mathbf{P}\bar{\mathbf{x}} + \bar{\mathbf{z}} = \mathcal{H}\mathbf{P}\bar{\mathbf{x}} + \bar{\mathbf{z}} = \bar{\mathbf{H}}\bar{\mathbf{x}} + \bar{\mathbf{z}}$$

- Can apply IF equalization to the aggregate channel [Domanovitz and Erez IEEEI12]

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But how to choose P to guarantee good performance?

- Large minimum distance for QAM translates to large SNR_{eff} for IF
- $\mathbf{P}$ should maximize $d^2_{\min}(\mathcal{H}\mathbf{P}, L)$ for the worst-case matrix $\mathbf{H}$
- This problem was extensively studied under the *linear dispersion space-time coding* framework
- "Perfect" linear dispersion codes guarantee that $d^2_{\min}(\mathcal{H}\mathbf{P}, L)$ grows appropriately with C_{WI}

Proving the Lower Bound

Theorem

If $\mathbf{P}$ generates a perfect linear dispersion code

$$\mathrm{SNR}d^2_{\min}(\mathcal{H}\mathbf{P},L) \geq \left[\delta_{\min}(\mathcal{C}^{\mathrm{ST}}_{\infty})^{\frac{1}{M}}2^{\frac{C_{\mathrm{WI}}}{M}} - 2M^2L^2\right]^+$$

for all channels matrices $\mathbf{H}$

Proof follows by using the properties of perfect codes and extending Tavildar and Vishwanath's proof for the approximate universality criterion

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Combining with the $\mathrm{SNR}_{\mathrm{eff}}$ lower bound

For precoded IF with a generating matrix **P** of a perfect ST "code"

$$\begin{aligned}\mathrm{SNR}_{\mathrm{eff}} &> \frac{1}{4M^4} \min_{L=1,2,\ldots} \left(L^2 + \mathrm{SNR} d^2_{\min}(\mathcal{H}\mathbf{P}, L)\right) \\ &\geq \frac{1}{8M^6} \delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}})^{\frac{1}{M}} 2^{\frac{C_{\mathrm{WI}}}{M}}\end{aligned}$$

Proving the Lower Bound

Since $R_{\mathrm{IF}} = M \log(\mathrm{SNR}_{\mathrm{eff}})$ we get the main result

For precoded IF with a generating matrix $\mathbf{P}$ of a perfect ST "code"

$$R_{IF} = M \log(\mathrm{SNR}_{\mathrm{eff}}) > C_{\mathrm{WI}} - \Gamma\left(\delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}}), M\right)$$

where $\Gamma\left(\delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}}), M\right) \triangleq \log \frac{1}{\delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}})} + 3M \log(2M^2)$

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Thanks for your attention!