# Successive Integer-Forcing and its Sum-Rate Optimality

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Joint work with Uri Erez and Bobak Nazer

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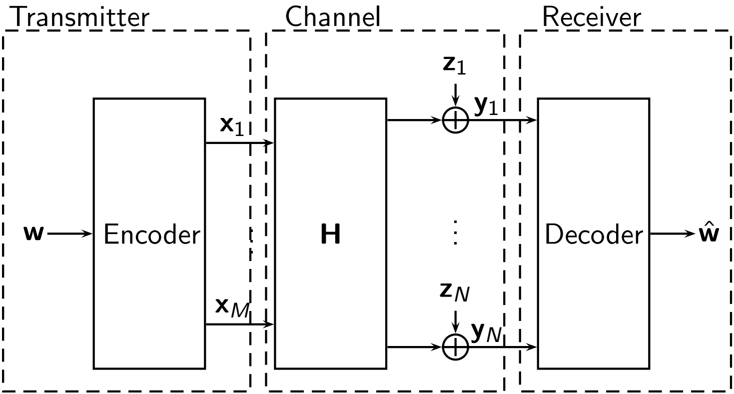
# Outline

- Review of standard successive cancelation decoding (through noise prediction)
- Review of integer-forcing equalization
- Successive integer-forcing
- Optimality of Korkin-Zolotarev reduction
- Asymmetric rates and sum-rate optimality

# The MIMO channel




$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$$

- $\mathbf{H} \in \mathbb{R}^{N\times M}$, $\mathbf{x} \in \mathbb{R}^{M\times 1}$ and $\mathbf{z} \sim \mathcal{N}(0, \mathbf{I}_{N\times N})$
- Power constraint is $\mathbb{E}\|\mathbf{x}_m\|^2 \leq \text{SNR}$ for $m = 1, \ldots, M$

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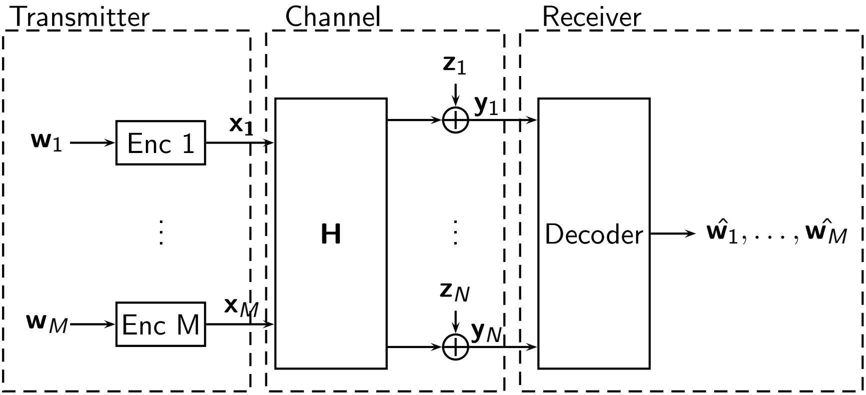


- We only consider BLAST schemes
  $\Longrightarrow$ All results are also valid for multiple access channels

# Sum rate optimality of SIC (via noise prediction)

- Assume each encoder uses an i.i.d. Gaussian codebook, such that $\mathbf{x}$ looks like $\mathcal{N}(\mathbf{0}, \text{SNR}\mathbf{I})$

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- Resulting effective channel is

$$\mathbf{y}_{\text{eff}} = \mathbf{B}\mathbf{y} = \mathbf{x} + \mathbf{e},$$

where $\mathbf{e} = \mathbf{B}\mathbf{y} - \mathbf{x} = (\mathbf{B}\mathbf{H} - \mathbf{I})\mathbf{x} + \mathbf{B}\mathbf{z}$ is a Gaussian vector with

$$\mathbf{K}_{\mathbf{ee}} = \text{SNR}(\mathbf{I} + \text{SNR}\ \mathbf{H}^T\mathbf{H})^{-1} = \text{SNR}\mathbf{G}\mathbf{G}^T$$

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- $\mathbf{e}$ can be written as $\mathbf{e} = \sqrt{\text{SNR}}\mathbf{G}\mathbf{w}$ where $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ and $\mathbf{G}$ is lower triangular matrix satisfying $(\mathbf{I} + \text{SNR}\ \mathbf{H}^T\mathbf{H})^{-1} = \mathbf{G}\mathbf{G}^T$

# Successive cancelation decoding via noise prediction

Equivalent channel after LMMSE estimation is

$$\mathbf{y}_{\text{eff}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{pmatrix} + \sqrt{\text{SNR}} \begin{pmatrix} g_{11} & 0 & \cdots & 0 \\ g_{21} & g_{22} & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ g_{M1} & g_{M2} & \cdots & g_{MM} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_M \end{pmatrix}$$

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- Decoding first stream from $y_{\text{eff},1} = x_1 + \sqrt{\text{SNR}} g_{11} w_1$ is possible if

$$R_1 < \frac{1}{2} \log \left( 1 + \frac{\text{SNR}}{\text{SNR} g_{11}^2} - 1 \right) = -\frac{1}{2} \log(g_{11}^2)$$

# Successive cancelation decoding via noise prediction

After decoding first stream, $w_1$ is also known and can be canceled from remaining streams

$$\mathbf{y}_{\text{eff}}^{(2)} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{pmatrix} + \sqrt{\text{SNR}} \begin{pmatrix} g_{11} & 0 & \cdots & 0 \\ \color{red}{0} & g_{22} & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \color{red}{0} & g_{M2} & \cdots & g_{MM} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_M \end{pmatrix}$$

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- Decoding second stream from $y_{\text{eff},2}^{(2)} = x_2 + \sqrt{\text{SNR}} g_{22} w_2$ is possible if

$$R_2 < \frac{1}{2} \log\left(1 + \frac{\text{SNR}}{\text{SNR} g_{22}^2} - 1\right) = -\frac{1}{2} \log(g_{22}^2)$$

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- Continuing in the same manner, each stream can be decoded if

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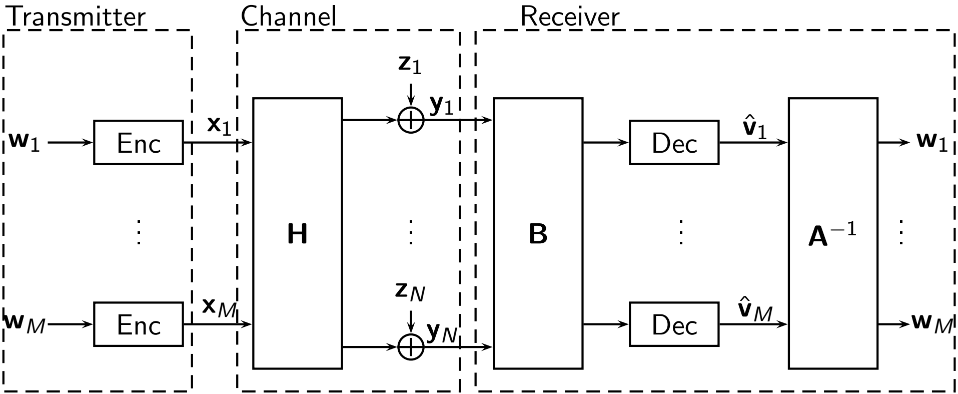
- Achievable sum-rate is

$$\begin{aligned}\sum_{m=1}^{M} R_m &= -\frac{1}{2}\sum_{m=1}^{M}\log\left(g_{mm}^2\right) \\ &= -\frac{1}{2}\log\left(\prod_{m=1}^{M} g_{mm}^2\right) \\ &= -\frac{1}{2}\log\det\left(\mathbf{G}\mathbf{G}^T\right) \\ &= \frac{1}{2}\log\det\left(\mathbf{I} + \mathrm{SNR}\mathbf{H}^T\mathbf{H}\right)\end{aligned}$$

# Integer-forcing - background




- Proposed by Zhan *et al.* ISIT2010

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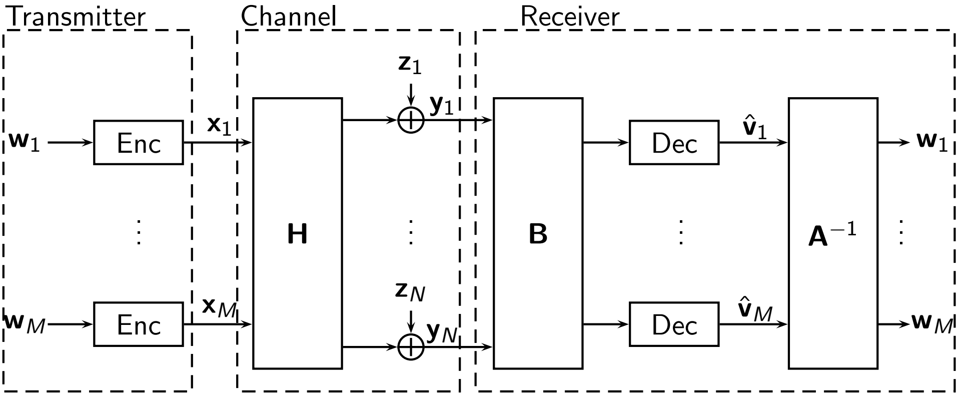


- Antennas transmit independent streams (BLAST).
- All streams are codewords from the **same linear code** with rate $R$.

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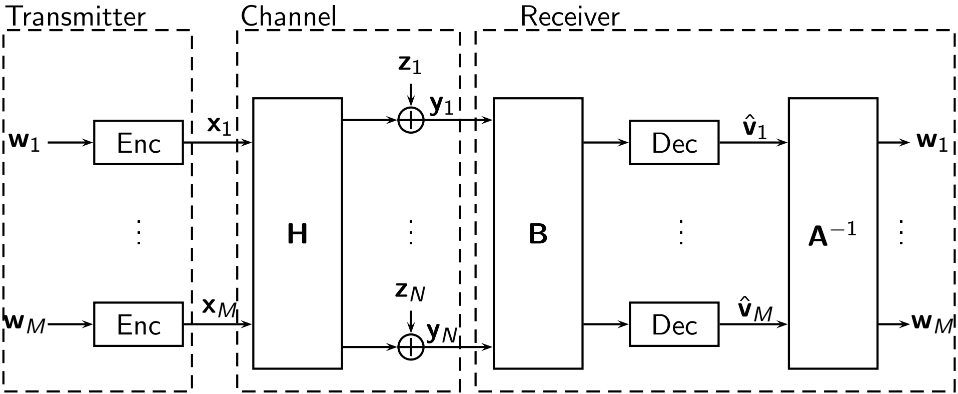


- Rather than estimating $\mathbf{x}$ from $\mathbf{y}$ as in standard linear equalizers, in IF $\mathbf{Ax}$ is estimated for some full-rank $\mathbf{A} \in \mathbb{Z}^{M \times M}$. LMMSE filter is

$$\mathbf{B} = \mathbf{A}\mathbf{H}^T \left( \mathrm{SNR}^{-1}\mathbf{I} + \mathbf{H}\mathbf{H}^T \right)^{-1}$$

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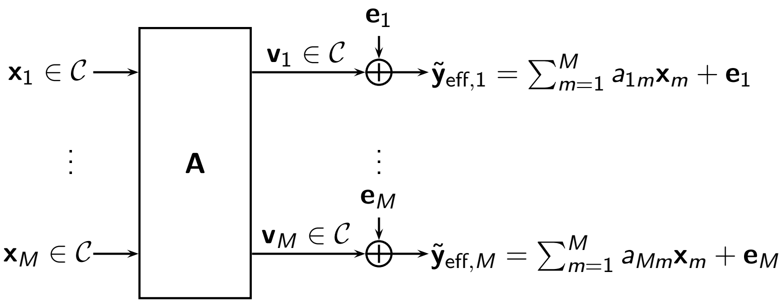


Effective channel is $\tilde{\mathbf{y}}_{\text{eff}} = \mathbf{A}\mathbf{x} + \mathbf{e}$

- A linear combination of codewords with integer coefficients is a codeword
  $\Longrightarrow$ Can decode the linear combinations - remove noise
  $\Longrightarrow$ Can solve noiseless linear combinations for transmitted streams

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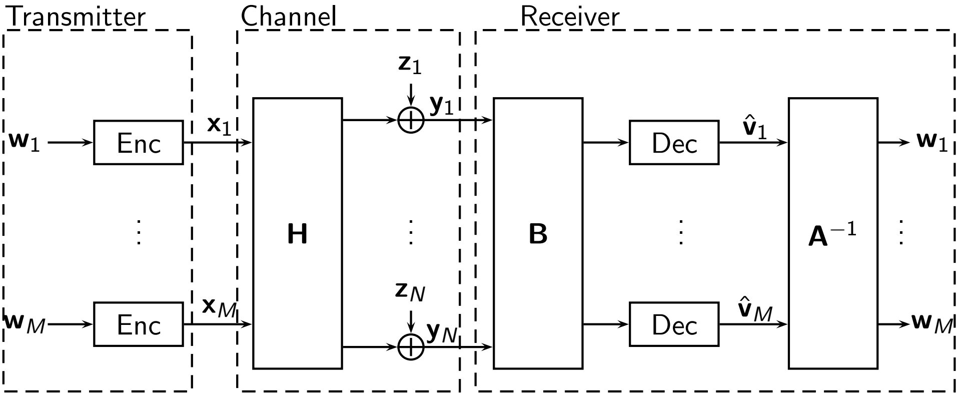


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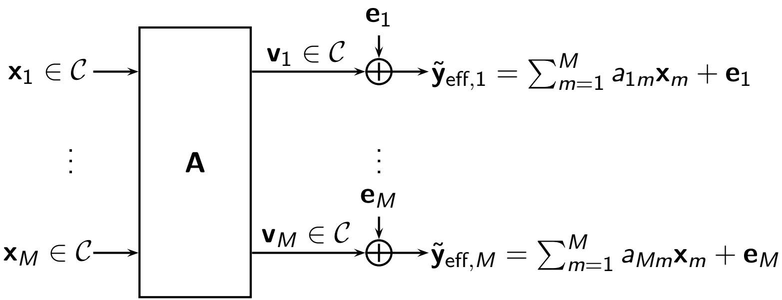

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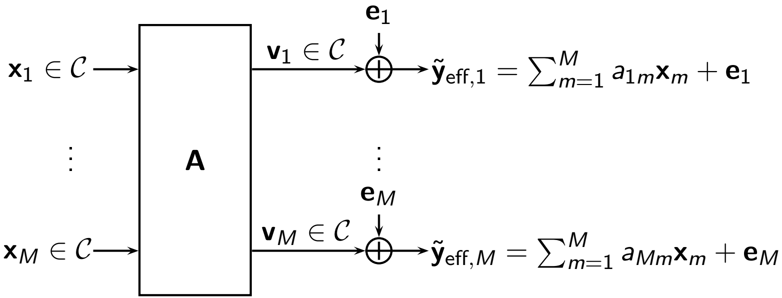


- For capacity achieving codebooks, the estimation errors behave like i.i.d. (in time) Gaussian RVs. The spatial covariance matrix is

$$\mathbf{K}_{\mathbf{ee}} = \text{SNR}\mathbf{A}(\mathbf{I} + \text{SNR}\ \mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T$$

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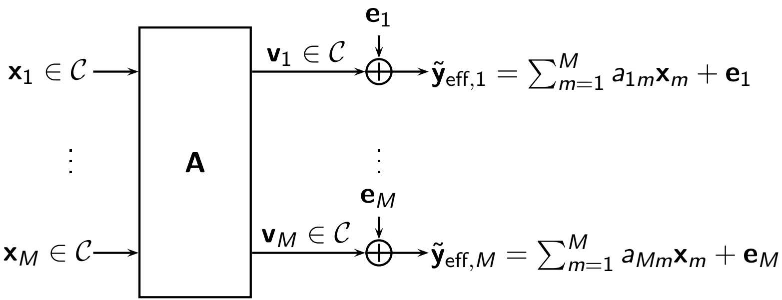


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Standard IF equalizer ignores the spatial correlations between estimation errors. Successive IF equalizer exploits them to increase rates

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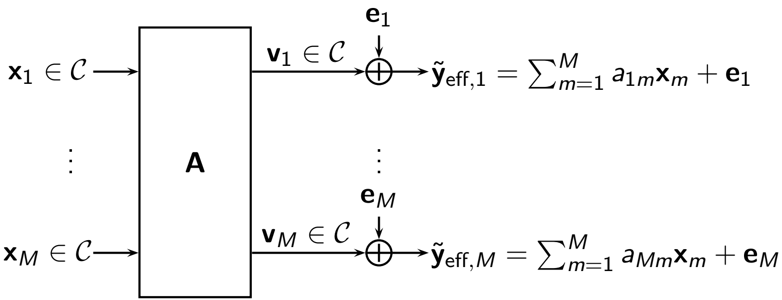


**Theorem (Nazer-Gastpar11IT)**

Each $\mathbf{v}_m$ can be decoded if $R < \frac{1}{2}\log\left(\frac{\text{SNR}}{\mathbf{K}_{\text{ee}}(m,m)}\right)$

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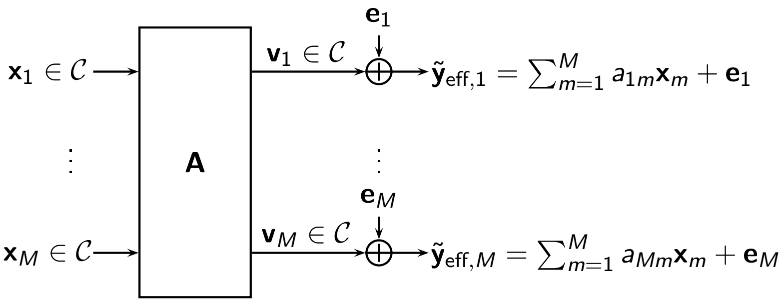


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All messages can be decoded if $R < \frac{1}{2}\log\left(\frac{\text{SNR}}{\max_m \mathbf{K}_{\text{ee}}(m,m)}\right)$

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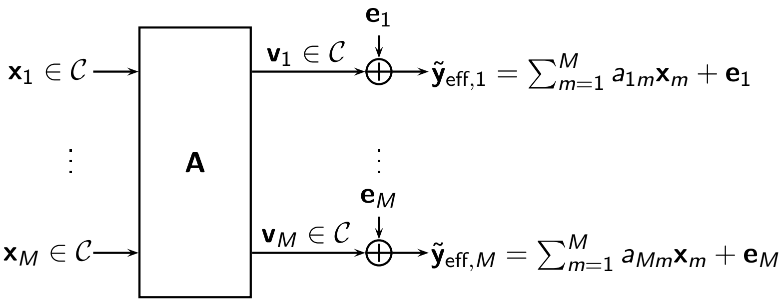

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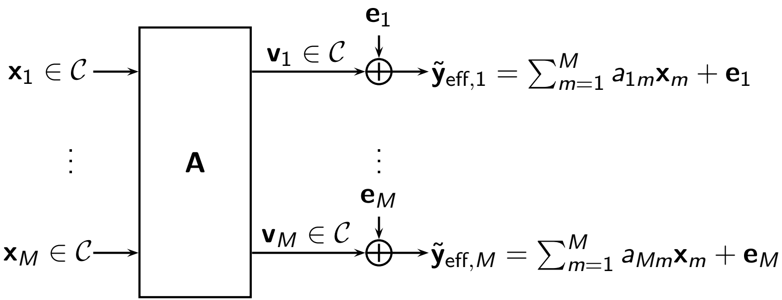


Let $\mathbf{L}$ be a lower triangular matrix such that $\text{SNR}\mathbf{L}\mathbf{L}^T = \mathbf{K}_{\mathbf{ee}}$. Using successive decoding we reduce the variance of $\mathbf{e}_m$ to $\text{SNR}\ell_{mm}^2$.

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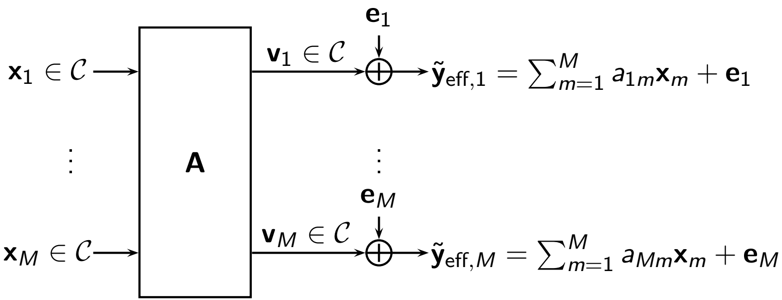


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How should we choose $\mathbf{A}$ for maximizing $R$?

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**Theorem**

The optimal $\mathbf{A}$ for successive integer-forcing can be found using Korkin-Zolotarev lattice basis reduction

$\Longrightarrow$ The optimal $\mathbf{A}$ always satisfies $|\mathbf{A}| = 1$ (unlike standard IF)

# Asymmetric rates

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Can this also be done for integer-forcing?

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- Assume that $M = 2$ (only two streams)
- First stream is taken from a linear code $\mathcal{C}_1$ with rate $R_1$
- Second stream is taken from a linear code $\mathcal{C}_2 \subset \mathcal{C}_1$ such that $R_2 < R_1$
- Both codes are over $\mathbb{Z}_5$
- Assume that $\mathbf{a}_1 = [2\ 3]^T$ and $\mathbf{a}_2 = [1\ 3]^T$

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The effective outputs after equalization are

$$\tilde{\mathbf{y}}_{\text{eff},1} = 2\mathbf{x}_1 + 3\mathbf{x}_2 + \mathbf{e}_1$$
$$\tilde{\mathbf{y}}_{\text{eff},2} = 1\mathbf{x}_1 + 3\mathbf{x}_2 + \mathbf{e}_2$$

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Reducing $\tilde{\mathbf{y}}_{\text{eff}}$ modulo 5 we get

$$\tilde{\mathbf{y}}_{\text{eff},1} = [2\mathbf{x}_1 + 3\mathbf{x}_2 + \mathbf{e}_1] \mod 5 = [\mathbf{v}_1 + \mathbf{e}_1] \mod 5$$
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- $\mathbf{v}_1 = [2\mathbf{x}_1 + 3\mathbf{x}_2] \mod 5 \in \mathcal{C}_1$
  $\Longrightarrow$ Can be decoded if $R_1$ sufficiently small w.r.t. $1/\ell_{11}^2$

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- $\mathbf{v}_2 = [1\mathbf{x}_1 + 3\mathbf{x}_2] \mod 5$ is also in $\mathcal{C}_1$
- Using the decoded $\mathbf{v}_1$ we can make it belong to $\mathcal{C}_2$
- $\mathcal{C}_2$ is sparser than $\mathcal{C}_1$
  $\implies$ Easier to decode

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After decoding $\mathbf{v}_1$ the receiver can add $2\mathbf{v}_1$ to $\tilde{\mathbf{y}}_{\text{eff},2}$ and reduce $\mod 5$

$$\begin{aligned}\tilde{\mathbf{y}}_{\text{eff},2}^{(2)} &= [1\mathbf{x}_1 + 3\mathbf{x}_2 + \mathbf{e}_2 + 2\mathbf{v}_1] \mod 5 \\ &= [(1+2\cdot 2)\mathbf{x}_1 + (3+2\cdot 3)\mathbf{x}_2 + \mathbf{e}_2] \mod 5 \\ &= [4\mathbf{x}_2 + \mathbf{e}_2] \mod 5\end{aligned}$$

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In addition $\mathbf{e}_1$ can be used to estimate $\mathbf{e}_2$

# Asymmetric rates

$[\mathbf{v}_2 + 2\mathbf{v}_1] \mod 5 = 4\mathbf{x}_2 \in \mathcal{C}_2$
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We also assumed $R_2 < R_1$
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If $\ell_{11}^2 \leq \ell_{22}^2$ this requirement is redundant

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If $\ell_{11}^2 \leq \ell_{22}^2$ we can encode one stream with rate $R_1 < -\frac{1}{2}\log\left(\ell_{11}^2\right)$ and the other stream with rate $R_2 < -\frac{1}{2}\log\left(\ell_{22}^2\right)$

# Sum-rate optimality of successive integer-forcing

If $\ell_{11}^2 \leq \cdots \leq \ell_{MM}^2$ the achievable sum-rate for successive integer-forcing is

$$\begin{aligned}
\sum_{m=1}^{M} R_m &= -\frac{1}{2}\sum_{m=1}^{M}\log\left(\ell_{mm}^2\right) \\
&= -\frac{1}{2}\log\left(\prod_{m=1}^{M}\ell_{mm}^2\right) \\
&= -\frac{1}{2}\log\det\left(\mathbf{L}\mathbf{L}^T\right) \\
&= -\frac{1}{2}\log\det\left(\mathbf{A}\left(\mathbf{I}+\mathsf{SNR}\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right) \\
&= \frac{1}{2}\log\det\left(\mathbf{I}+\mathsf{SNR}\mathbf{H}^T\mathbf{H}\right) - \log|\det(\mathbf{A})|
\end{aligned}$$

# Sum-rate optimality of successive integer-forcing

If $\ell_{11}^2 \leq \cdots \leq \ell_{MM}^2$ the achievable sum-rate for successive integer-forcing is

$$\begin{aligned}\sum_{m=1}^{M} R_m &= -\frac{1}{2}\sum_{m=1}^{M}\log\left(\ell_{mm}^2\right)\\ &= -\frac{1}{2}\log\left(\prod_{m=1}^{M}\ell_{mm}^2\right)\\ &= -\frac{1}{2}\log\det\left(\mathbf{L}\mathbf{L}^T\right)\\ &= -\frac{1}{2}\log\det\left(\mathbf{A}\left(\mathbf{I}+\mathsf{SNR}\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right)\\ &= \frac{1}{2}\log\det\left(\mathbf{I}+\mathsf{SNR}\mathbf{H}^T\mathbf{H}\right)-\log|\det(\mathbf{A})|\end{aligned}$$

There is always an optimal $\mathbf{A}$ with $|\det(\mathbf{A})| = 1$, so the sum-rate is optimal

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## Why is this important in closed-loop?

For MIMO it is not very important

For MAC each stream belongs to a different user and symmetry is often desired

# Gaussian MAC with nested linear codes - IF rate region

Gaussian two-user MAC $\mathbf{y} = 1\mathbf{x}_1 + \sqrt{2}\mathbf{x}_2 + \mathbf{z}$ at SNR $= 15$dB