A VC-Dimension-Based Outer Bound on the Zero-Error Capacity of the Binary Adder Channel

Or Ordentlich
Joint work with Ofer Shayevitz

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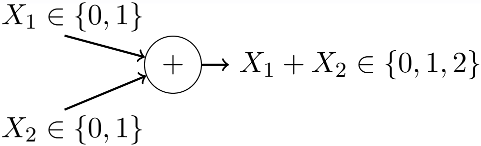
The Binary Adder Channel



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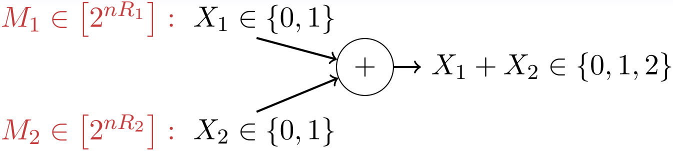

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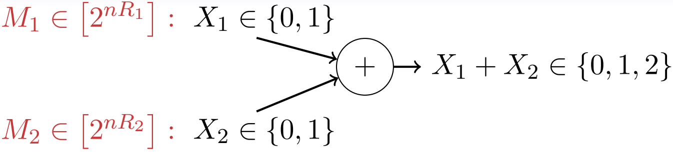

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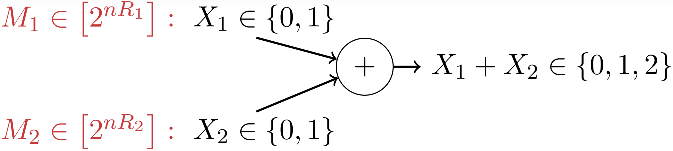

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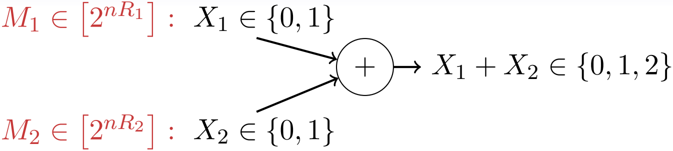

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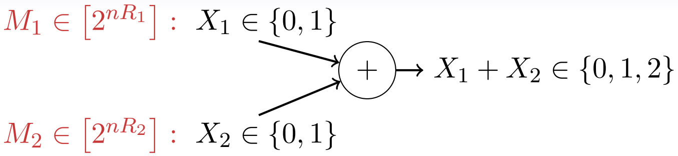
Shannon Capacity Region (vanishing error)

- Simple MAC channel... the region is

$$R_1 \leq 1$$
$$R_2 \leq 1$$
$$R_1 + R_2 \leq 1.5$$

The Binary Adder Channel



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- No errors allowed (no collisions)
- All sums are different

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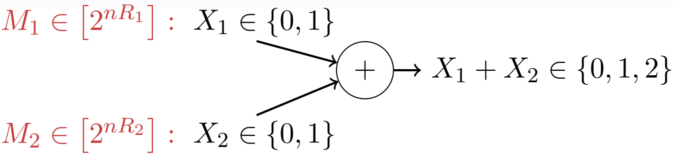

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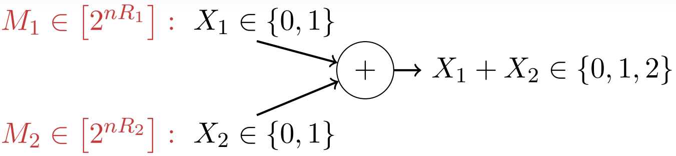

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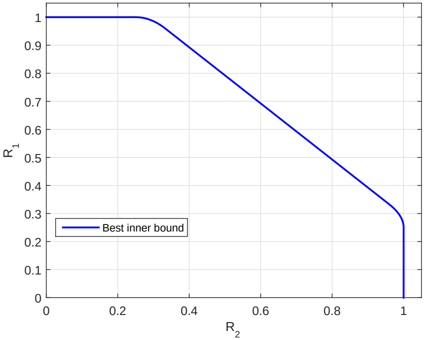
- **In this talk: Outer bound improved**

KNOWN BOUNDS



[*Lindström '69*], [*Kasami & Lin '76*], [*Weldon '78*], [*Peterson & Costello '79*], [*Khachatrian '82*], [*van Tilborg '83*], [*Kasami et al '83*], [*van der Braak & van Tilborg '85*], [*Guo & Watanabe '91, '92*], [*Blake '94*], [*Ahlswede & Balakirsky '99*], [*Mattas & Ostergard '05*]

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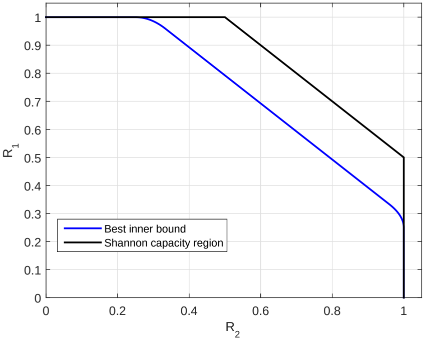

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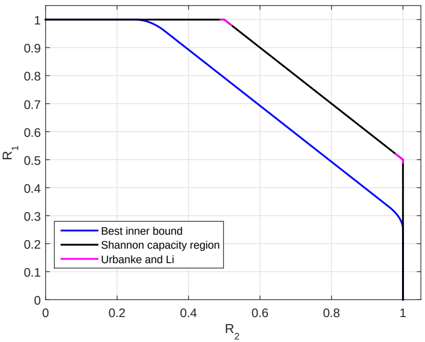

[*Urbanke & Li 1998*]

- For $R_1 = 1$: $0.25 \leq R_2 < 0.49216$ [*Kasami et al '83*], [*Urbanke & Li '98*]

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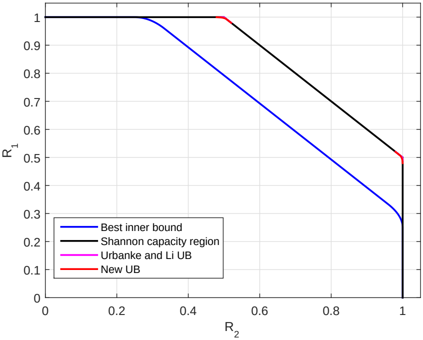

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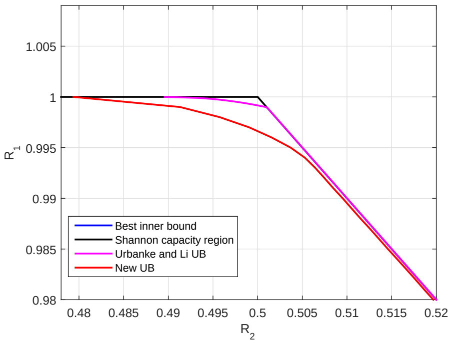

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Projection and Shattering

- Let $\mathbf{c} \in \{0,1\}^n$
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$\mathcal{C} = \{101100, 011101, 100110, 000111\} \Rightarrow \mathcal{S} = \{1,2\}$ not shattered

Theorem [*Weldon 78*]

Let $(\mathcal{C}_1, \mathcal{C}_2)$ be a pair of zero-error codebooks for the BAC with cardinalities $(2^{nR_1}, 2^{nR_2})$. If $\mathcal{C}_1$ shatters a set of coordinates S with cardinality $|S| = n\alpha$, then

$$R_2 \leq (1-\alpha)\log 3$$

Proof:

- Choose any $\mathbf{c}_2 \in \mathcal{C}_2$
- By shatterdness, there exists $\mathbf{c_1} \in \mathcal{C}_1$ such that $\mathbf{c}_1(S) = \overline{\mathbf{c}_2(S)}$, i.e.,

$$(\mathbf{c}_1 + \mathbf{c}_2)(S) = (1 \ldots, 1)$$

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- Weldon stopped here, but what can be said in general?

VC Dimension [*Vapnik-Chervonenkis 1971*]

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If the VC dimension of $\mathcal{C}$ is d, then

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Corollary

If $|\mathcal{C}| = 2^{n(R+\varepsilon)}$ then there is a shattered set S with $|S| \geq nh^{-1}(R)$.

Background & Motivation

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- Weaknesses:
  - We assumed only one S-complement for each $\mathbf{c}_2 \in \mathcal{C}_2$
    Weak lower bound on # of S-complement pairs (2^{nR_2})
  - We disregarded the sumset structure outside S
    Weak upper bound on # of S-complement pairs ($3^{|\bar{S}|}$)

HIGHER ORDER VC DIMENSION

k-shattering

$S \subseteq [n]$ is *k-shattered by* $\mathcal{C} \subseteq \{0,1\}^n$, if $\mathcal{C}(S)$ contains all $2^{|S|}$ possible vectors, each with multiplicity at least k

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$\mathcal{C} = \{101100, 011101, 100110, 000111\}$, $\mathcal{S} = \{6\}$ 2-shattered

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kth-order VC dimension

The kth-order *VC dimension* of a codebook $\mathcal{C} \subseteq \{0,1\}^n$ is the cardinality of the largest subset k-shattered by $\mathcal{C}$

Higher Order VC Dimension

Lemma ("soft" Sauer-Perles-Shelah lemma)

If the kth-order VC dimension of $\mathcal{C} \subseteq \{0,1\}^n$ is $d-1$, then

$$|\mathcal{C}| \le \sum_{t=1}^{t^*} \binom{n}{t} + \binom{n}{t^*} \sum_{t=t^*+1}^{n} \frac{\binom{t^*}{d}}{\binom{t}{d}}$$

where t^* is the smallest integer t satisfying $\binom{n-d}{t-d} \ge k$ if such an integer exists, and $t^* = n$ otherwise.

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- $\mathcal{O}(n/d)$-tight for a Hamming Ball of radius t^*

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where t^* is the smallest integer t satisfying $\binom{n-d}{t-d} \geq k$ if such an integer exists, and $t^* = n$ otherwise.

- $\mathcal{O}(n/d)$-tight for a Hamming Ball of radius t^*

Corollary

If $|\mathcal{C}| = 2^{n(R+\varepsilon)}$ then for any $0 \leq \alpha \leq h^{-1}(R)$ there exists S with $|S| \geq n\alpha$ that is $2^{n\beta}$-shattered by $\mathcal{C}$, where

$$\beta = (1-\alpha) \cdot h\left(\frac{h^{-1}(R) - \alpha}{1-\alpha}\right)$$

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- Example: $S = \{1, 2\}$

$\mathcal{C}_1$:	$\mathcal{C}_2$:
0010110	1000110
1111001	0111101
1101011	1001010
0110111	1111101
0000101	0100110
1010010	1011101
0111000	0000000
⋮	⋮

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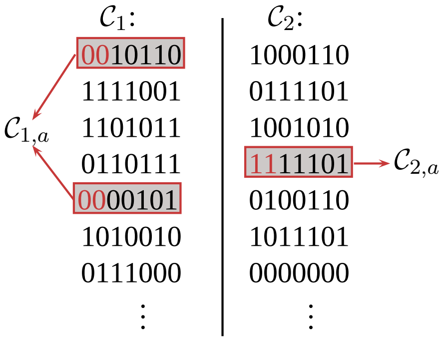

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1111001	0111101
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0110111	1111101 ($\in \mathcal{C}_{2,a}$)
0000101 ($\in \mathcal{C}_{1,a}$)	0100110
1010010	1011101
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⋮	⋮

$\forall \mathbf{c}_{1,a} \in \mathcal{C}_{1,a}, \mathbf{c}_{2,a} \in \mathcal{C}_{2,a}$:

```
  00*****
+ 11*****
---------
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In addition: $|\mathcal{C}_{1,a}| \geq 2^{n\beta}$

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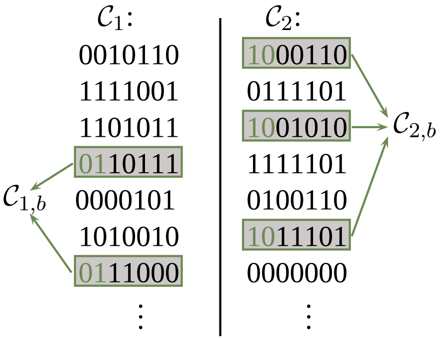

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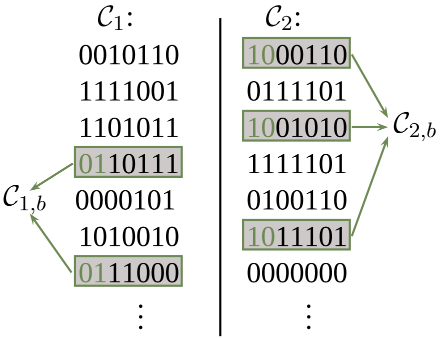

$\forall \mathbf{c}_{1,b} \in \mathcal{C}_{1,b}, \mathbf{c}_{2,b} \in \mathcal{C}_{2,b}$:

$$\begin{array}{r} 01{*}{*}{*}{*}{*} \\ +\;10{*}{*}{*}{*}{*} \\ \hline 11{*}{*}{*}{*}{*} \end{array}$$

In addition: $|\mathcal{C}_{1,b}| \geq 2^{n\beta}$

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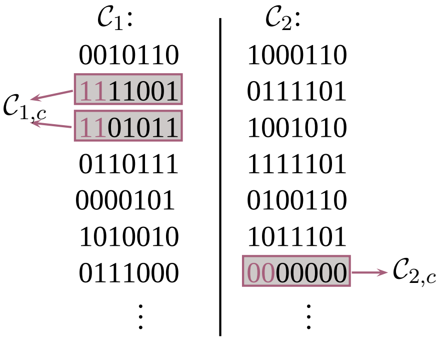

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- We get $\{\mathcal{C}_{1,i}, \mathcal{C}_{2,i}\}_{i=1}^{2^{n\alpha}}$ with $|\mathcal{C}_{1,i}| = 2^{n\beta}$ and $|\mathcal{C}_{2,i}| \approx 2^{n(R_2-\alpha)}$ for every i

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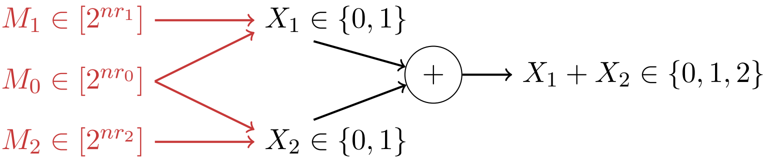

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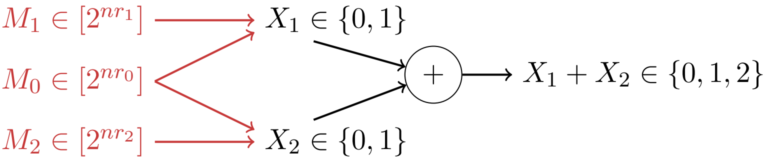

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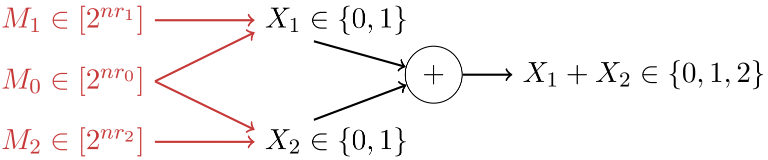

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$$(\mathbf{c}_{1,i} + \mathbf{c}_{2,i})(S) = (1, \ldots, 1)$$

- The $n\alpha$ coordinates in S can be discarded!

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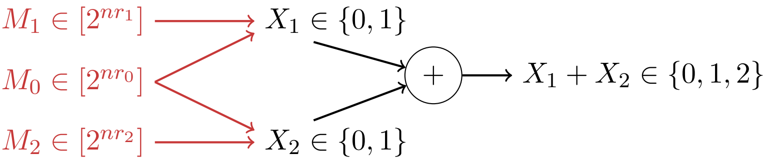

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The BAC with a Common Message

A reduction lemma

If (R_1, R_2) are in the BAC zero-error capacity region, then for any $0 < \alpha < h^{-1}(R_1)$

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The Binary Adder With a Common Message

Theorem [*Slepian-Wolf 1973*], [*Willems 1982*]

The Shannon capacity region of the BAC with a common message, is the closure of the union of all rate triplets satisfying

$$
\begin{aligned}
r_1 &\leq H(X_1|U) \\
r_2 &\leq H(X_2|U) \\
r_1 + r_2 &\leq H(X_1 + X_2|U) \\
r_\Sigma = r_0 + r_1 + r_2 &\leq H(X_1 + X_2)
\end{aligned}
$$

for some $P_U P_{X_1|U} P_{X_2|U}$, where $|\mathcal{U}| \leq 4$.

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for some $P_U P_{X_1|U} P_{X_2|U}$, where $|\mathcal{U}| \leq 3$.

- Still difficult - 7 parameters to optimize...
- Need to upper bound $r_\Sigma(r_0, r_1)$ analytically!

Lemma (Sum-Rate Bound)

If (r_0, r_1, r_2) is achievable then there is some $\eta \in [0, \frac{1}{2}]$ s.t.

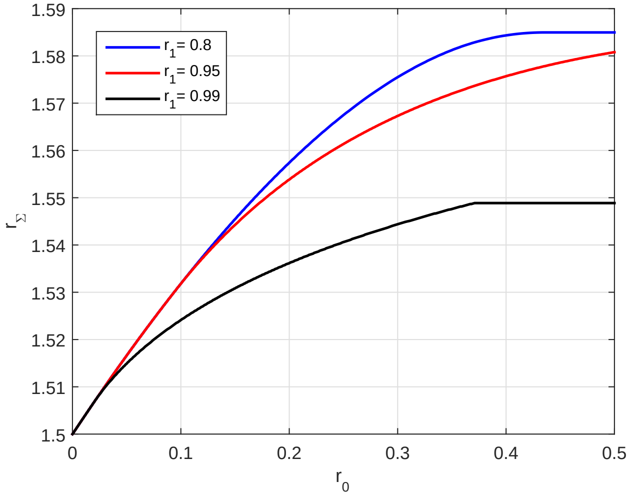
$$r_\Sigma \leq \max_{h^{-1}(r_1) \leq \eta \leq \frac{1}{2}} \min\left(L(\eta), J(h^{-1}(r_1), \eta) + r_0\right)$$

where

$$L(\eta) \stackrel{\text{def}}{=} h(\eta) + 1 - \eta$$

$$J(p, \eta) \stackrel{\text{def}}{=} \begin{cases} 2h\left(\frac{1}{2}\left(1 - \sqrt{1 - 2\eta}\right)\right) - \eta & \eta \geq p \star p \\ 2h\left(\frac{1}{2}\left(1 - \frac{1-\eta-p\star p}{\sqrt{1-2(p\star p)}}\right)\right) \\ \quad -\frac{1}{2}\left(1 - \frac{(1-\eta-p\star p)^2}{1-2(p\star p)}\right) & \eta < p \star p \end{cases}$$

The Binary Adder With a Common Message



Tying the Loose Ends

Theorem (Outer bound for the Zero-error Capacity Region)

Let

$$r_\Sigma(r_0, r_1) \triangleq \max_{h^{-1}(r_1) \le \eta \le \frac{1}{2}} \min\{L(\eta),\ J(h^{-1}(r_1), \eta) + r_0\}$$

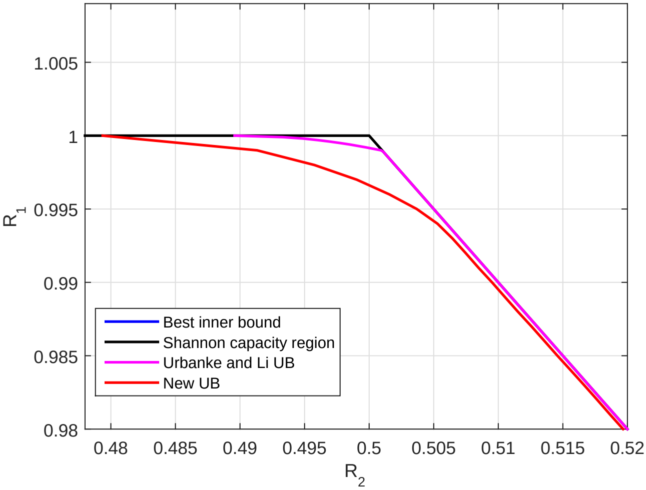
Then any zero-error achievable rate pair (R_1, R_2) satisfies

$$R_2 < \min_{0 \le \alpha \le h^{-1}(R_1)} (1-\alpha)\left(r_\Sigma\left(\frac{\alpha}{1-\alpha}, \Gamma(R_1, \alpha)\right) - \Gamma(R_1, \alpha)\right)$$

where

$$\Gamma(R_1, \alpha) \triangleq h\left(\frac{h^{-1}(R_1) - \alpha}{1-\alpha}\right)$$

Tying the Loose Ends



Summary and Discussion

- New outer bound on BAC zero-error capacity region
- We introduced the notion of kth order VC-dimension and proved an analog of Sauer's Lemma
- Our bounding technique combined this combinatorial notion with network information theoretic arguments

- Weaknesses of our bound
  - The lower bound on $\#$ of S-complement pairs is valid for any pair $(\mathcal{C}_1, \mathcal{C}_2)$ (not just zero-error pairs)
  - We lower bounded the number of S-complement pairs for *any* k-shattered set in $\mathcal{C}_1$, but there are *many such sets*.
- Our technique may be applicable to other zero-error problems