Mergers and Collusion in All-Pay Auctions and Crowdsourcing Contests

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All-pay auctions

Bidders bid and pay their bid to the auctioneer

Auction winner is one which submitted the highest bid
Why all-pay auctions?

Explicit all-pay auctions are rare, but implicit ones are extremely common:

- Competition for patents between firms
- Crowdsourcing competitions (e.g., Netflix challenge, TopCoder, etc.)
- Hiring employees
- Employee competition (“employee of the month”)
Auctioneer types

“sum profit”
- Gets the bids from all bidders – regardless of their winning status
- E.g., “employee of the month”

“max profit”
- Gets only the winner’s bid. Other bids are, effectively, “burned”
- E.g., hiring an employee
All-pay auction equilibrium

All bidders give the object in question a value of 1

A single symmetric equilibrium – for \( n \) bidders:

\[
F_n(x) = x^{\frac{1}{n-1}}
\]

\[
f_n(x) = \frac{x^{\frac{2-n}{n-1}}}{n-1}
\]

Baye, Kovenock, de Vries
All-pay auction equilibrium
bidder properties

Expected utility: 0

Utility variance: \( \frac{3n^2 - 5n + 2}{n(2n - 1)(3n - 2)} \)

Expected bid: \( \frac{1}{n} \)

Bid variance: \( \frac{1}{2n - 1} - \frac{1}{n^2} \)

Baye, Kovenock, de Vries
All-pay auction equilibrium
auctioneer properties

Regular all-pay

Sum profit
expected profit:

\[ \frac{n}{2n - 1} - \frac{1}{n} \]

Sum profit
profit variance:

Max profit
expected profit:

\[ \frac{n}{2n - 1} \]

Max profit
profit variance:

\[ \frac{n(n - 1)^2}{(3n - 2)(2n - 1)^2} \]

Baye, Kovenock, de Vries
Example no collusion case

3 bidders

Bidders’ c.d.f is $\sqrt{x}$ and the expected bid is $\frac{1}{3}$, with variance of $\frac{4}{45}$. Expected profit is 0 with variance of $\frac{2}{15}$.

Sum profit auctioneer has expected profit of 1 with variance of $\frac{4}{15}$.

Max profit auctioneer has expected profit of $\frac{3}{5}$ with variance of $\frac{12}{175}$.

Baye, Kovenock, de Vries
$k$ bidders (out of the total $n$) collaborate, having a joint strategy. All other bidders are aware of this.
Merger properties

Equilibrium remains the same – but with smaller $n$

**Bidder**
- Expected Utility: 0
- Utility variance:
- Expected bid:
- Bid variance:

**Sum Profit**
- Expected profit: 1
- Profit variance:

**Max Profit**
- Expected profit:
- Profit variance:
Example no collusion case

3 bidders

Bidders’ c.d.f is $\sqrt{x}$ and the expected bid is $\frac{1}{3}$, with variance of $\frac{4}{45}$. Expected profit is 0 with variance of $\frac{2}{15}$.

Sum profit auctioneer has expected profit of 1 with variance of $\frac{4}{15}$.

Max profit auctioneer has expected profit of $\frac{3}{5}$ with variance of $\frac{12}{175}$.
3 bidders, 2 of them merged

Bidders’ c.d.f is uniform, and the expected bid is $\frac{1}{2}$, with variance of $\frac{1}{12}$. Expected profit is 0 with variance of $\frac{1}{6}$.

Sum profit auctioneer has expected profit of 1 with variance of $\frac{1}{6}$.

Max profit auctioneer has expected profit of $\frac{2}{3}$ with variance of $\frac{1}{18}$. 
$k$ bidders (out of the total $n$) collaborate, having a joint strategy. Other bidders are not aware of this and continue to pursue their previous strategies.
Collusion colluders

Colluders have a pure, optimal strategy

\[ b^* = \left( \frac{n - k}{n - 1} \right)^{\frac{n-1}{k-1}} \]

Producing an expected profit of:

\[ \left( \frac{n - k}{n - 1} \right)^{\frac{n-1}{k-1}} \left( \frac{k - 1}{n - 1} \right) \]

Colluders’ profit per colluder increases as number of colluders grows

Profit variance:

\[ \left( \frac{n - k}{n - 1} \right)^{\frac{n-k}{k-1}} - \left( \frac{n - k}{n - 1} \right)^{\frac{2(n-k)}{k-1}} \]
Collusion auctioneers

Sum profit: \( \frac{n - k}{n} + \left( \frac{n - k}{n - 1} \right)^{\frac{n-1}{k-1}} \)

Max profit: \( \frac{n - k}{2n - k - 1} \left( 1 + \left( \frac{n - k}{n - 1} \right)^{\frac{2(n-k)}{k-1}} \right) \)

For large enough \( n \) exceed non-colluding profits
Utility for non-colluding bidders is:

\[
\frac{k}{n(n - k)} - \left(\frac{n-k}{n-1}\right)\frac{n-k}{k-1}
\]

For large enough \(k\) (e.g., \(\frac{n}{2}\)) this expression is positive. I.e., non-colluders profit from collusion.

If a non-colluder discovers the collusion, best to bid a bit above colluders.
Example **no collusion case**

3 bidders

Bidders’ c.d.f is $\sqrt{x}$ and the expected bid is $\frac{1}{3}$, with variance of $\frac{4}{45}$. Expected profit is 0 with variance of $\frac{2}{15}$.

Sum profit auctioneer has expected profit of 1 with variance of $\frac{4}{15}$.

Max profit auctioneer has expected profit of $\frac{3}{5}$ with variance of $\frac{12}{175}$. 
Example merger case

3 bidders, 2 of them merged

Bidders’ c.d.f is uniform, and the expected bid is $\frac{1}{2}$, with variance of $\frac{1}{12}$. Expected profit is 0 with variance of $\frac{1}{6}$.

Sum profit auctioneer has expected profit of 1 with variance of $\frac{1}{6}$.

Max profit auctioneer has expected profit of $\frac{2}{3}$ with variance of $\frac{1}{18}$. 

Collusion (collaboration private knowledge)
Example collusion case

3 bidders, 2 of them collude

One bidder has c.d.f of \( \sqrt{x} \) (expected bid of \( \frac{1}{3} \)), colluders bid \( \frac{1}{4} \). Colluders’ expected profit is \( \frac{1}{4} \), while the non-colluder expected profit is \( \frac{1}{6} \).

Sum profit auctioneer expected profit only \( \frac{7}{12} \).

Max profit auctioneer has expected profit of \( \frac{10}{24} \).
Future directions

- Adding bidders’ skills to model
- Detecting collusions by other bidders
- Designing crowdsourcing mechanisms less susceptible to collusion
- Adding probability to win based on effort
The End

Thanks for listening!