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Matrix Multiplication, A Little Faster

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Goals

Improve the practical performance of fast matrix multiplication algorithms.

1. We reduce the number of arithmetic operations by a constant factor
2. We reduce the number of data transfers within memory hierarchy by a constant factor

Introduction

Strassen's algorithm (1969), was the first sub-cubic matrix multiplication algorithm. Winograd (1971) improved the leading coefficient of its complexity from 7 to 6. Can we do better?

Theorem (Probert 1976). *Any Strassen-like algorithm with 2x2 base case and 7 multiplications requires at least 15 additions.*

Strassen-Winograd's algorithm was believed to be optimal due to this bound.

Theorem (Karstadt & Schwartz 2017). *There is a Strassen-like algorithm with 2x2 base case and 7 multiplications requires at least 12 additions.*

Our theorem seems to implicitly contradict Probert's lower bound. However, this bound assumes that the input and output are represented in the standard basis. We extend Probert's lower pound to account for alternative bases.

Theorem (Karstadt & Schwartz 2017). *Irrespective of input/output bases, a Strassen-like algorithm with 2x2 base case and 7 multiplications requires at least 12 additions.*

Our generalization of Probert's lower bound shows our algorithm to be optimal for matrix multiplication with 2x2 base case and with 7 multiplications.

Encoding/Decoding matrices

Any bi-linear algorithm which uses t multiplications can be described by encoding/decoding matrices $\langle U, V, W\rangle$:

$$U \in R^{t\times n\cdot m},\ V \in R^{t\times m\cdot k},\ W \in R^{t\times n\cdot k}$$

Such that $\forall A \in R^{n\cdot m},\ B \in R^{m\cdot k}$

$$ALG\left(A, B\right) = W^T\left(\left(U\cdot A\right)\odot\left(V\cdot B\right)\right)$$

where $\cdot$ is matrix multiplication and $\odot$ is element-wise vector product (Hadamard product).

Our Method

Strassen-like $\langle n, m, k; t\rangle$-algorithms are block-recursive algorithms. Defined by $n\times m$, $m\times k$, and $n\times k$ base case for the linear part, and t multiplications.
Bodrato [2] used a method of intermediate representation of 2×2 matrices for repeated squaring and for chain matrix multiplication computations. We extend this method to alternative basis Strassen-like matrix multiplication.

Alternative basis Strassen-like $\langle n, m, k; t\rangle_{\phi,\psi,\upsilon}$-algorithms take input $\phi\left(A\right)$, $\psi\left(B\right)$ and output $\upsilon\left(A\cdot B\right)$.

Lemma (Karstadt & Schwartz 2017). *Let R be a ring, and let ϕ, ψ, υ be automorphisms of $R^{n\cdot m}$, $R^{m\cdot k}$, $R^{n\cdot k}$ (respectively). $\langle U, V, W\rangle$ are encoding/decoding matrices of an $\langle n, m, k; t\rangle_{\phi,\psi,\upsilon}$-algorithm if and only if $\langle U\phi, V\psi, W\upsilon^{-T}\rangle$ are encoding/decoding matrices of an $\langle n, m, k; t\rangle$-algorithm*

We compute over alternative bases where:

- Transformation between the standard and our alternative basis can be done in $O\left(n^2\log n\right)$ time, which is asymptotically negligible.
- A Strassen-like algorithm in our alternative basis uses fewer additions/subtractions.

Optimal $\langle 2,2,2;7\rangle_{\phi,\psi,\upsilon}$-algorithm

$$U_{opt} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ -1 & 1 & 0 & 0\\ 0 & -1 & 0 & 1 \end{pmatrix} V_{opt} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\\ 0 & 1 & -1 & 0\\ -1 & 1 & 0 & 0 \end{pmatrix} W_{opt} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & -1 & 0 & -1\\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \{\phi,\psi\}_{opt} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -1 & 1\\ 0 & 0 & -1 & 1\\ 0 & 1 & 0 & 1 \end{pmatrix} \upsilon_{opt}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 0 & 1\\ 0 & -1 & 1 & 1 \end{pmatrix}$$

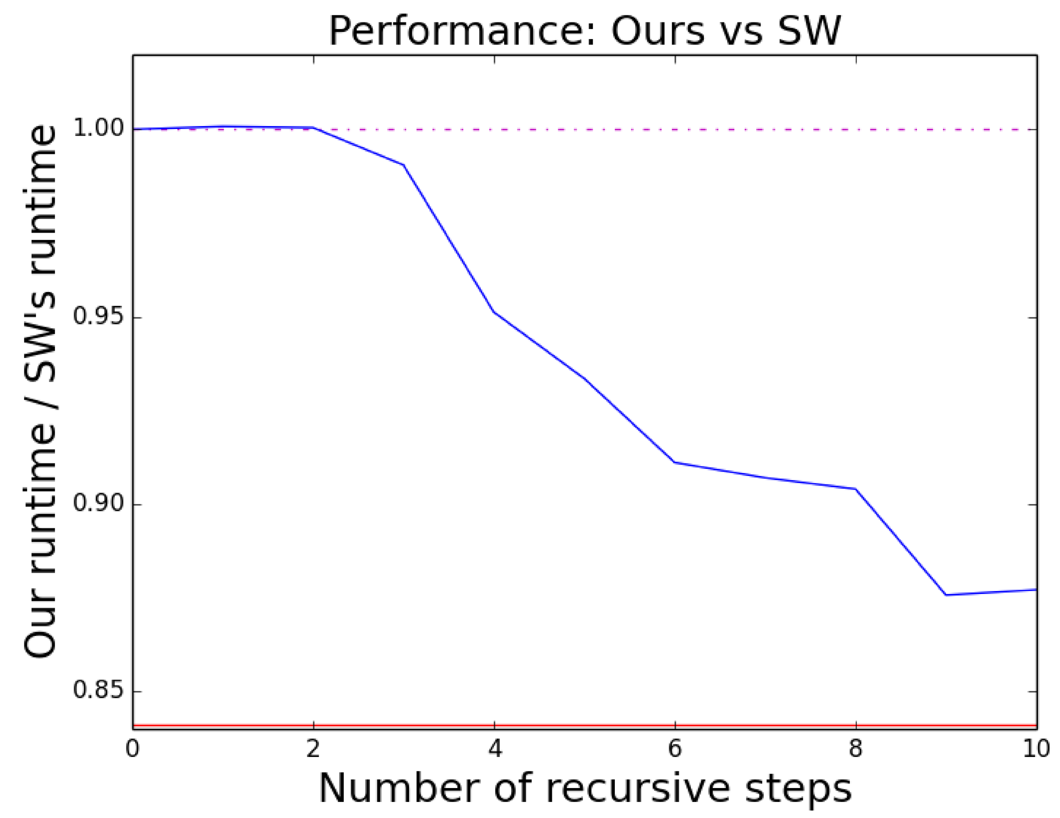
Algorithm 1: Encoding/Decoding and basis transformation matrices for our $\langle 2,2,2;7\rangle_{\phi,\psi,\upsilon} - algorithm$

Comparing our $\langle 2,2,2;7\rangle_{\phi,\psi,\upsilon}$-algorithm to Strassen-Winograd's

Algorithm	Arithmetic Complexity	I/O-Complexity (with a cache of size M)
Strassen [6]	$7n^{\log_2 7} - 6n^2$	$6\cdot\left(\frac{\sqrt{3}\cdot n}{\sqrt{M}}\right)^{\log_2 7}\cdot M - 18n^2 + 3M$
Strassen-Winograd [8]	$6n^{\log_2 7} - 5n^2$	$5\cdot\left(\frac{\sqrt{3}\cdot n}{\sqrt{M}}\right)^{\log_2 7}\cdot M - 15n^2 + 3M$
Ours	$5n^{\log_2 7} - 4n^2 + 3n^2\log_2 n$	$4\cdot\left(\frac{\sqrt{3}\cdot n}{\sqrt{M}}\right)^{\log_2 7}\cdot M - 12n^2 + 3n^2\cdot\log_2\left(\sqrt{2}\cdot\frac{n}{\sqrt{M}}\right) + 5M$

Table 2: Complexity of $\langle 2,2,2;7\rangle$-algorithms

Due to its lower leading coefficient of 5 instead of 6, our $\langle 2,2,2;7\rangle_{\phi,\psi,\upsilon}$-algorithm asymptotically performs 16.6% less arithmetic operations than Strassen-Winograd's. Preliminary benchmark results indicate that our algorithm achieves an improvement close to theory even on modestly sized input ($N = 32768$) with few cores ($P = 6$), and outperforms Strassen-Winograd's algorithm.
We expect the 20% improvement in I/O-complexity to play a greater role on larger inputs with more cores.



Alternative Basis Strassen

Given an $\langle n_0, m_0, k_0; t\rangle_{\phi,\psi,\upsilon}$-algorithm (RBA), and its corresponding $\phi \in GL_{n_0\cdot m_0}\left(R\right)$, $\psi \in GL_{m_0\cdot k_0}\left(R\right)$, $\upsilon \in GL_{n_0\cdot k_0}\left(R\right)$

Input: $A \in R^{n\times m}$, $B \in R^{m\times k}$
Output: $C \in R^{n\times k}$ such that $C = A\cdot B$

1: **function** $ABS(A, B)$
2: $\quad\tilde{A} = \phi(A)$ $\quad\triangleright O\left(nm\cdot\log_{n_0\cdot m_0}(nm)\right)$
3: $\quad\tilde{B} = \psi(B)$ $\quad\triangleright O\left(mk\cdot\log_{m_0\cdot k_0}(mk)\right)$
4: $\quad\tilde{C} = RBA(\tilde{A}, \tilde{B})$
5: $\quad C = \upsilon^{-1}(\tilde{C})$ $\quad\triangleright O\left(nk\cdot\log_{n_0\cdot k_0}(nk)\right)$
6: $\quad$**return** C

Basis transformations are block-recursive, i.e., given ϕ_1 we define:

$$\left(\psi_{k+1}\left(A\right)\right)_{i,j} = \psi_k\left(\psi_1\left(A\right)\right)_{i,j}$$

Further Applications

Algorithm	Prev. Coefficient	New Coefficient	Saves
$\langle 3,2,3;15\rangle$[1]	15.06	7.94	47.2%
$\langle 2,3,4;20\rangle$[1]	9.96	7.46	25.1%
$\langle 2,2,2;7\rangle$[8]	6	5	16.6%
$\langle 6,3,3;40\rangle$[5]	55.63	9.39	83.1%

Table 1: A Sample of Alternative Basis Algorithms

Finding optimal bases for Strassen-like algorithms reduces to the Matrix Sparsification problem, which is NP-Hard [3]. However, our need is for fixed base case sizes.
For algorithms with a base case larger than Strassen-Winograd's, the search space gets quite big. We utilized computer aided search and found several alternative basis variants of known Strassen-like algorithms.

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