An Elementary Construction of Constant-Degree Expanders*

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Abstract

We describe a short and easy to analyze construction of constant-degree expanders. The construction relies on the replacement product, applied by [14] to give an iterative construction of bounded-degree expanders. Here we give a simpler construction, which applies the replacement product (only twice!) to turn the Cayley expanders of [4], whose degree is polylog n, into constant degree expanders. This enables us to prove the required expansion using a new simple combinatorial analysis of the replacement product (instead of the spectral analysis used in [14]).

1 Introduction

All graphs considered here are finite, undirected and may contain self-loops and parallel edges. Expanders are graphs, which are simultaneously sparse, yet highly connected, in the sense that every cut contains (relatively) many edges. In this note we mostly work with the notion of *edge-expansion*. A d-regular graph $G = (V, E)$ is a δ-edge-expander (δ-expander for short) if for every set $S \subseteq V$ of size at most $\frac{1}{2}|V|$ there are at least $\delta d|S|$ edges connecting S and $\overline{S} = V \setminus S$, that is, $e(S, \overline{S}) \geq \delta d|S|$. For brevity, we say that a graph is an $[n, d, \delta]$-expander if it is an n-vertex d-regular δ-expander. Expanders are some of the most widely used objects in theoretical computer science, and have also found many applications in other areas of computer-science and mathematics. See the survey of Hoory et. al. [9] for a discussion of several applications and references. Another widely used notion of expansion is based on algebraic properties of a matrix representation of the graph. Let $G = (V, E)$ be an n-vertex d-regular graph, and let A be the adjacency matrix of G, that is, the $n \times n$ matrix, with $A_{i,j}$ being the number of edges between i and j. It is easy to see that 1^n is an eigenvector of A with eigenvalue d, and that this is the only eigenvector with this eigenvalue iff G is connected. We denote by $\lambda_2(G)$ the second largest eigenvalue of A. It is easy to see that $\lambda_2(G) = \max_{0 \neq x \perp 1^n} \langle Ax, x\rangle / \langle x, x\rangle$. The following is a well known relation between the expansion of G and $\lambda_2(G)$.

Theorem 1 ([2], [3] ,[6]) *Let G be a δ-expander with adjacency matrix A and let $\lambda_2 = \lambda_2(G)$ be the second largest eigenvalue of A. Then,* $\frac{1}{2}(1 - \lambda_2/d) \leq \delta \leq \sqrt{2(1 - \lambda_2/d)}$.

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Our construction uses only the first simple inequality, but for completeness, we include a very short proof of the second direction of this theorem in the appendix.

The most useful expanders are those with constant degree. A priori, it is not clear that constant-degree expanders even exist. Pinsker [13] established their existence.

Theorem 2 ([13]) *There exists a fixed $\delta > 0$, such that for any $d \geq 3$ and even integer n, there is an $[n, d, \delta]$-expander, which is d-edge-colorable* [1].

One way to prove the above is to take a random d-regular bipartite graph. In most applications, however, one needs to efficiently construct constant degree expanders explicitly. There are two notions of constructibility of d-regular expanders. The first (weaker) notion requires the n-vertex graph to be constructible in polynomial time in its size. The second (stronger) notion requires that given a vertex v and $i \in [d]$ it would be possible to generate the i^{th} neighbor of v in time $\mathrm{poly}(\log n)$. Such an expander is said to be *fully explicit*. In applications, where one needs to use the entire graph, it is often enough to use the weaker notion. However, in such cases (e.g. in certain reductions) one frequently needs to be able to construct a graph of a given size n. It has been observed by many that to this end it is enough to be able to construct graphs of size $\Theta(n)$ (e.g., one can take a cn-vertex expander and join groups of c vertices to get an n-vertex expander with positive expansion). In other cases, where one needs only part of the expander (e.g., when performing a random walk on a large expander) one usually needs the stronger notion of fully explicitness. However, in these cases it is usually enough to be able to construct an expander of size $\mathrm{poly}(n)$, as what we are interested in is actually the logarithm of the size of the graph.

Margulis [11] and Gabber and Galil [7] were the first to efficiently construct constant degree expanders. Following was a sequence of works that culminated in the construction of Lubotzky, Phillips and Sarnak [10] and Margulis [12] of Ramanujan Graphs. These constructions rely (directly or indirectly) on estimations of the second largest eigenvalue of the graphs, and some of them, rely on deep mathematical results. A simpler, iterative construction was given by Reingold, Vadhan and Wigderson [14]. This construction also relies on proving the expansion of the graphs by estimating their eigenvalues, and is the first construction of constant degree expanders with relatively elementary analysis. Additional algorithms for constructing bounded-degree expanders appear in [1] and [5], but these should be viewed as algorithms rather than explicit constructions, and in particular are not fully explicit in the sense described above.

Our construction is based on the replacement product of two graphs G and H, which is one of the most natural ways of combining two graphs. We start by defining this basic operation.

Definition 1.1 *Let G be a D-regular D-edge-colorable graph on n vertices and let H be a d-regular graph on D vertices. Suppose G is already equipped with a proper D-edge-colorings. The* replacement product *$G \circ H$ is the following $2d$-regular graph on nD vertices: We first replace every vertex v_i of G with a cluster of D vertices, which we denote $C_i = \{v_1^i, \ldots, v_D^i\}$. For every $1 \leq i \leq n$ we put a copy of H on C_i by connecting v_p^i to v_q^i if and only if $(p, q) \in E(H)$. Finally, for every $(p, q) \in E(G)$, which is colored t, we put d parallel edges between v_t^p and v_t^q.*

Note that if H is d-edge-colorable then $G \circ H$ is $2d$-edge colorable: simply color the copies of H within each set C_i using colors $1, \ldots, d$. As the edges between the sets C_i form d parallel copies of a

[1] That is, one can assign its edges d colors such that edges incident with the same vertex are assigned distinct colors.

perfect matching on the vertices of $G \circ H$, we can color any set of d parallel edges using the colors $d+1, \ldots, 2d$. Already in the 80's, Gromov [8] has analyzed the effect of (a slight variant of) this operation on the spectral properties of graphs. Reingold, Vadhan and Wigderson [14] considered the above variant, and showed, via a reduction to their algebraic analysis of the zig-zag product, that if two graphs are expanders then so is their product. Their argument is based on analyzing $\lambda(G \circ H)$ as a function of $\lambda(G)$ and $\lambda(H)$. We analyze the replacement product *directly* via an elementary combinatorial argument.

Theorem 3 *Suppose E_1 is an $[n, D, \delta_1]$-expander and E_2 is a $[D, d, \delta_2]$-expander. Then, $E_1 \circ E_2$ is an $[nD, 2d, \frac{1}{80}\delta_1^2\delta_2]$-expander.*

The proof of Theorem 3 is very simple; we show that $e(X, \overline{X})$ has either many edges within the clusters C_i or between them. Our main result is a new construction of constant-degree expanders. The main idea can be summarized as follows: a simple special case of one of the results of [4] gives a construction of $[n, O(\log^2 n), \frac{1}{4}]$-expanders. To get expanders with constant degree we construct such an $[n, O(\log^2 n), \frac{1}{4}]$-expander and then apply the replacement product with another similar expander in order to reduce the degree to $O(\sqrt{\log n})$. We now find a constant degree expander of size $O(\sqrt{\log n})$, using exhaustive search, and apply a final replacement product to get a constant degree. Note that here we do not care much about the fact that the replacement product decreases the edge-expansion as we only apply it twice. A suitable choice of parameters gives the following construction, whose analysis relies solely on the easy part of Theorem 1, a simple special case of the result of [4] and on the elementary analysis of the replacement product (Theorem 3).

Theorem 4 (Main Result) *There exists a fixed $\delta > 0$ such that for any integer $q = 2^t$ and for any $q^4/100 \leq r \leq q^4/2$ there is a polynomial time constructible $[q^{4r+12}, 12, \delta]$-expander.*

For completeness we prove all the necessary ingredients, thus obtaining a short and self-contained construction of constant-degree expanders. It is easy to see that given n, Theorem 4 can be used to construct an m-vertex expander with $n \leq m = O(n \log n)$. The construction and its analysis appear in the following section. In Section 3 we observe that simple variants of Theorem 4 give a construction with $\Theta(n)$ vertices and a construction which is fully explicit.

2 The Construction

Let us start by describing the special case of [4] that suffices for our purposes. For any $q = 2^t$ and $r \in \mathbb{N}$, we define a graph $LD(q, r)$ as follows. The vertices are all elements of $\mathbb{F}_q^{r+1}$, which can be thought of as all strings $(a_0, \ldots, a_r) \in \mathbb{F}_q^{r+1}$. A neighbor of a vertex a is indexed by an element $(x, y) \in \mathbb{F}_q^2$. In this notation neighbor (x, y) of vertex $a = (a_0, \ldots, a_r)$ is $a + y \cdot (1, x, x^2, \ldots, x^r)$. $LD(q, r)$ is clearly a q^2-regular graph on q^{r+1} vertices. It is also q^2-edge-colorable as we can color the edges indexed (x, y) using the "color" (x, y) (note that this is well defined as addition and subtraction are identical in $\mathbb{F}_{2^t}$). The following result is a special case of the result of [4]:

Theorem 5 ([4]) *For any $q = 2^t$ and integer $r < q$ we have $\lambda_2(LD(q, r)) \leq rq$.*

Note that the above theorem, together with the left inequality of Theorem 1, imply that if $r \le q/2$ then $LD(q,r)$ is a $[q^{r+1}, q^2, \frac{1}{4}]$-expander. We first prove our main result based on Theorems 3, 5 and the left inequality of Theorem 1. We then prove these three results.

We next show a combinatorial analysis of the replacement product. The analysis here is not tight and could be improved. Indeed, we do not try to present the strongest possible bound, but rather to give one with a simple proof. Note that it suffices for our purpose, as we apply it only a constant number of times.

Proof of Theorem 4: Given integers q and $\frac{q^4}{100} \le r \le \frac{q^4}{2}$, we start by enumerating all 3-regular graphs on q^2 vertices until we find one which is a δ-expander and 3-edge colorable (one exists by Theorem 2). This step can clearly be carried out in time $q^{O(q^2)}$. Denote by E_1 the expander we find and define $E_3 = LD(q^4, r)$, $E_2 = LD(q, 5)$ and set $E_4 = E_3 \circ (E_2 \circ E_1)$ to be our final graph. As E_1, E_2 and E_3 are $[q^2, 3, \delta]$, $[q^6, q^2, \frac{1}{4}]$ and $[q^{4r+4}, q^8, \frac{1}{4}]$ expanders respectively, E_4 is a $[q^{4r+12}, 12, \delta']$-expander for some absolute constant δ' (here we rely on Theorem 3). Moreover, given E_1 one can easily compute E_4 in time polynomial [2] in the size of E_4. As $r \ge q^4/100$, E_4 is of size at least $q^{q^4/10}$, thus the first step of finding E_1 also takes time polynomial in the size of E_4, as needed. ■

Proof of Theorem 3: Let $E_3 = E_1 \circ E_2$ and consider any set X of vertices in E_3 of size at most $\frac{1}{2}nD$. Note that we can view the vertex set of E_3 as composed of n clusters of vertices $C_1, \ldots, C_n$, each of size D. Our goal is to show that there are at least $\frac{1}{80}\delta_1^2\delta_2 \cdot 2d|X|$ edges leaving X. We consider two cases. Either many of the vertices of X are in clusters C_i which are sparsely populated by X, in which case many edges are leaving X within the clusters C_i due to the expansion properties of E_2. Or there are many of the vertices of X which reside in densely populated clusters C_i, in which case there are many edges leaving X between the clusters, due to the expansion properties of E_1.

Set $X_i = X \cap C_i$, let $I' \subseteq [n]$ be the set of indices of the sets X_i, whose size is at most $(1 - \frac{1}{4}\delta_1)D$ and let $I'' = \{1, \ldots, n\} \setminus I'$. We first consider the contribution of the sets X_i with $i \in I'$. As E_2 is a δ_2-expander, there are at least $\frac{1}{4}\delta_1\delta_2 d|X_i|$ edges connecting X_i and $C_i \setminus X_i$. Partition X into two sets X' and X'' according to I' and I'' as follows: $X' = \bigcup_{i \in I'} X_i$ and $X'' = \bigcup_{i \in I''} X_i$. By the above, the number of edges connecting X' and $\overline{X}$ is at least $\frac{1}{4}\delta_1\delta_2 d|X'|$. If $|X'| \ge \frac{1}{10}\delta_1|X|$ then we are done, as this means that there are at least $\frac{1}{80}\delta_1^2\delta_2 \cdot 2d|X|$ edges connecting X and its complement $\overline{X}$.

Suppose then that $|X'| \le \frac{1}{10}\delta_1|X|$, implying that $|X''| \ge (1 - \frac{1}{10}\delta_1)|X|$. We now consider the contribution of the edges leaving the sets C_i. As the sets X_i with $i \in I''$ have size at least $(1 - \frac{1}{4}\delta_1)D$ we infer that $|X''|/D \le |I''| \le |X''|/(1 - \frac{1}{4}\delta_1)D$. In particular, as $|X''| \le |X| \le \frac{1}{2}nD$ we have $|I''| \le \frac{2}{3}n$. Therefore, as E_1 is an $[n, D, \delta_1]$-expander, there is a set of edges M, where $|M| \ge \frac{1}{2}\delta_1 D|I''|$, connecting the vertices of I'' with the vertices of I'. Let us now consider the corresponding $d|M| \ge \frac{1}{2}\delta_1 dD|I''|$ edges in the graph E_3. These edges connect vertices from $\bigcup_{i \in I'} C_i$ with vertices from $\bigcup_{i \in I''} C_i$. As each X_i with $i \in I''$ is of size at least $(1 - \frac{1}{4}\delta_1)D$, we infer that at most $\frac{1}{4}\delta_1 dD|I''|$ of these $d|M|$ edges connect a vertex in $C_i \setminus X_i$ with a vertex of $\bigcup_{i \in I'} C_i$. Therefore, there are at least $\frac{1}{4}\delta_1 dD|I''|$ edges connecting $\bigcup_{i \in I''} X_i$ with the vertices of $\bigcup_{i \in I'} C_i$. The number of these $d|M|$ edges that connect vertices from $\bigcup_{i \in I''} C_i$ with vertices of X' is clearly at most $d|X'|$. As we have $|X'| \le \frac{1}{10}\delta_1|X| \le \frac{1}{6}\delta_1 D|I''|$ we infer that there are at most $\frac{1}{6}\delta_1 dD|I''|$ such edges. We conclude that at least $\frac{1}{12}\delta_1 dD|I''|$ edges connect vertices of $\bigcup_{i \in I''} X_i$ (that belong to X) with vertices

[2] Note that when constructing E_2 and E_3 we need representations of $\mathbb{F}_q$ and $\mathbb{F}_{q^4}$. These representations can be found using exhaustive search in time poly(q^4) that is much smaller than the size of E_4 and thus negligible.

of $\bigcup_{i\in I'} C_i \setminus X_i$ (that belong to $\overline{X}$). As $|I''| \geq |X''|/D$ and $|X''| \geq \frac{1}{2}|X|$ this means that there are at least $\frac{1}{48}\delta_1 2d|X|$ edges connecting X and $\overline{X}$, as needed. ■

Proof of Theorem 5: Set $\mathbb{F} = \mathbb{F}_{2^t}$, $n = 2^{t(r+1)}$ and let M be the $n \times n$ adjacency matrix of $LD(2^t, r)$. Let $L : \mathbb{F} \to \{0,1\}$ be any surjective linear map [3]. Let us describe the eigenvectors of M over $\mathbb{R}$. We will use elements of $\mathbb{F}^{r+1}$ in order to "name" these vectors as well as to "name" entries of these vectors. For every sequence $a = (a_0, \ldots, a_r) \in \mathbb{F}^{r+1}$, let v_a be the vector, whose b^{th} entry (where $b \in \mathbb{F}^{r+1}$) satisfies $v_a(b) = (-1)^{L(\sum_{i=0}^r a_i b_i)}$. It is easy to see that the vectors $\{v_a\}_{a\in\mathbb{F}^{r+1}}$ are orthogonal, therefore these are the only eigenvectors of M. Clearly, $v_a(b+c) = v_a(b)v_a(c)$ for any $b, c \in \mathbb{F}^{r+1}$. Let us show that v_a is indeed an eigenvector and en-route also compute its eigenvalue.

$$(Mv_a)(b) = \sum_{c\in\mathbb{F}^{r+1}} M_{bc} \cdot v_a(c) = \sum_{x,y\in\mathbb{F}} v_a(b + y(1, x, ..., x^r)) = \left(\sum_{x,y\in\mathbb{F}} v_a(y, yx, ..., yx^r)\right) \cdot v_a(b) \ .$$

Therefore $\lambda_a = \sum_{x,y\in\mathbb{F}} v_a(y, yx, ..., yx^r)$ is the eigenvalue of v_a. Set $p_a(x) = \sum_{i=0}^r a_i x^i$ and write

$$\lambda_a = \sum_{x,y\in\mathbb{F}} (-1)^{L(y\cdot p_a(x))} = \sum_{\{x,y\in\mathbb{F}\ :\ p_a(x)=0\}} (-1)^{L(y\cdot p_a(x))} + \sum_{\{x,y\in\mathbb{F}\ :\ p_a(x)\neq 0\}} (-1)^{L(y\cdot p_a(x))} \ .$$

If $p_a(x) = 0$, then $(-1)^{L(y\cdot p_a(x))} = 1$ for all y, thus such an x contributes q to λ_a. If $p_a(x) \neq 0$ then $y \cdot p_a(x)$ takes on all values in $\mathbb{F}$ as y varies, and hence $(-1)^{L(y\cdot p_a(x))}$ varies uniformly over $\{-1, 1\}$ implying that these x's contribute nothing to λ_a. Therefore, when $a = 0^n$ we have $\lambda_a = q^2$. Otherwise, when $a \neq 0^n$, p_a has at most r roots, and therefore $\lambda_a \leq rq$. ■

Proof of left inequality of Theorem 1: Let A be the adjacency matrix of G and note that as A is symmetric we have $\lambda_2 = \max_{0\neq x\perp 1^n} \langle xA, x\rangle/\langle x, x\rangle$. For a set $S \subseteq V(G)$ let x_S be the vector satisfying $x_i = 1$ when $i \in S$ and $x_i = 0$ otherwise, and note that $\langle x_S A, x_S\rangle = 2e(S)$ and $\langle x_S A, x_{\overline{S}}\rangle = e(S, \overline{S})$. Set $x = |\overline{S}| \cdot x_S - |S| \cdot x_{\overline{S}}$ and note that $x \perp 1^n$. Therefore,

$$\lambda_2(|S| + |\overline{S}|)|S||\overline{S}| = \lambda_2\langle x, x\rangle \geq \langle xA, x\rangle = 2|S|^2 e(\overline{S}) + 2|\overline{S}|^2 e(S) - 2|S||\overline{S}|e(S, \overline{S}). \qquad (1)$$

As G is d-regular we have $e(S) = \frac{1}{2}(d|S| - e(S, \overline{S}))$ and $e(\overline{S}) = \frac{1}{2}(d|\overline{S}| - e(S, \overline{S}))$. Plugging this into (1), solving for $e(S, \overline{S})$ and using $|S| \leq n/2$, we complete the proof by inferring that

$$e(S, \overline{S}) \geq (d - \lambda_2)|S||\overline{S}|/n \geq \frac{1}{2}(d - \lambda_2)|S| \ .$$

3 Concluding Remarks

Expanders on $\Theta(n)$ vertices: Let us first show how to apply Theorem 4 in order to construct for every large n, an expander on $\Theta(n)$ vertices. Let N_t be the set of integers of the form q^{4r+12} where $q = 2^t$ and $q^4/100 \leq r \leq q^4/2$. By Theorem 4 we can generate an expander of size n for every $n \in \bigcup_{t=1}^\infty N_t$ in time poly(n). Note that for every $t \geq 2$ we have

$$\max\{N_t\} = q^{4\cdot\frac{q^4}{2}+12} = 2^{\lg q(2q^4+12)} > 2^{(\lg q+1)(\frac{64}{100}q^4+12)} = (2q)^{4\cdot\frac{(2q)^4}{100}+12} = \min\{N_{t+1}\} \ .$$

[3] For example, if we view the elements of $\mathbb{F}$ as element of $\{0,1\}^t$ then we can define $L(a_0, a_1, \ldots, a_{t-1}) = a_0$.

Therefore for every $n \geq 4^{4 \cdot \frac{4^4}{100}+12}$ there exists a t such that $n \in [\min\{N_t\}, \max\{N_t\}]$. Hence, for every such n there exists a $q = 2^t$ and $\frac{q^4}{100} \leq r_0 \leq \frac{q^4}{2}$ such that $n/q^4 \leq q^{4r_0+12} \leq n$.

Now, given n let $q = 2^t$ and $q^4/100 \leq r_0 \leq q^4/2$ be such that $n/q^4 \leq q^{4r_0+12} \leq n$ (as guaranteed by the previous paragraph). We start by using Theorem 4 to construct a $[q^{4r_0+12}, 12, \delta]$-expander E satisfying $n/q^4 \leq q^{4r_0+12} \leq n$. If $n/32 \leq q^{4r_0+12}$ we return E. Otherwise set $t = \lfloor n/16q^{4r_0+12} \rfloor < q^4$ and use exhaustive search to find a 6-regular expander E' on $12t$ vertices (which exists by Theorem 2). This step takes time $q^{O(q^4)}$, which is polynomial in the size of E because $|E| \geq q^{\frac{1}{25}q^4}$ as $r \geq q^4/100$. We now replace every edge of E with t parallel edges to get a $[q^{4r_0+12}, 12t, \delta]$-expander E''. We then define $E'' \circ E'$ to be the final 12-regular graph on m vertices with $n/2 \leq m \leq n$.

Fully explicit expanders: We now show that for every t we can construct a fully explicit $[2^{t\lfloor 2^t/t \rfloor}, d, \delta]$-expander for some constants $d, \delta > 0$. Thus, for every n we can construct such an expander of size $n \leq m \leq n^2$. We use the previous argument to find an expander of size $2^{2t} \leq m \leq c2^{2t}$. As noted in Section 1 we can then turn it into a constant degree expander E_1 of size precisely 2^{2t}. This step takes time $2^{O(t)}$. It is useful to "name" the vertices of E_1 using pairs of elements of $\mathbb{F}_{2^t}$. Set $E_2 = LD(2^t, \lfloor 2^t/t \rfloor - 3)$ and define $E_3 = E_2 \circ E_1$ as the final constant degree expander on $2^{t\lfloor 2^t/t \rfloor}$ vertices. To see that E_3 is fully explicit, note that we can view a vertex of $LD(q, r)$ as composed of $r+1$ elements of $\mathbb{F}_q$. Therefore, a vertex of $E_3 = E_2 \circ E_1$ can be viewed as $r+1 = \lfloor 2^t/t \rfloor - 2$ elements $(a_0, \ldots, a_r)$ of $\mathbb{F}_{2^t}$ (representing a vertex of E_2) and another pair of elements x, y of $\mathbb{F}_{2^t}$ (representing a vertex of E_1). Suppose the degree of E_1 is d' in which case the degree of E_3 is $2d'$. Given $r+3$ elements $(a_0, \ldots, a_r, x, y)$ of $\mathbb{F}_{2^t}$ and $i \in [2d']$ we do the following. If $1 \leq i \leq d'$ we return $(a_0, \ldots, a_r, x', y')$, where (x', y') is the i^{th} neighbor of vertex (x, y) in E_1. We can do so by generating E_1 from scratch in time $2^{O(t)}$. If $d'+1 \leq i \leq 2d'$, we return the vertex $(a'_0, \ldots, a'_r, x, y)$, where $a'_i = a_i + yx^i$. To do so we use a representation of $\mathbb{F}_{2^t}$ that we find using exhaustive search in time $2^{O(t)}$. We finally note that one can easily adopt our arguments to get *space efficient* variants of our constructions. We omit the details.

Edge expansion close to $\frac{1}{2}$: The expanders we constructed have a positive edge expansion. However, by applying Theorem 1 it is easy to see that for every ϵ we can raise the graphs we construct to an appropriate power to get edge-expansion $\frac{1}{2} - \epsilon$. In fact, to get edge-expansion $\frac{1}{2} - \epsilon$ one needs the degree to be poly$(1/\epsilon)$.

Eigenvalue gap: As we have mentioned before all the previous constructions of bounded-degree expanders did so via constructing a graph, whose second eigenvalue is bounded away from d. Theorem 1 implies that if G is an $[n, d, \delta]$-expander then its second largest eigenvalue is at most $d(1 - \frac{1}{2}\delta^2)$. As we can construct expanders with edge expansion close to $\frac{1}{2}$, these graphs have second largest eigenvalue at most roughly $\frac{7}{8}d$. By adding loops and raising the resulting graphs to an appropriate power one can get expanders in which all eigenvalues are, in absolute value, at most some fractional power of the degree of regularity.

Expanders with smaller degree: The expanders we construct have constant degree larger than 3. In order to get 3-regular expander one can take any constant degree d-regular expander and apply a replacement product with a cycle of length d. Definition 1.1 implies that the new degree is 4, but it is easy to see that when d is a constant we do not have to duplicate each edge of the "large" graph d times, as keeping a single edge guarantees a positive expansion. This way we can get a 3-regular expander, which is clearly the smallest possible degree of regularity.

References

[1] M. Ajtai, Recursive construction for 3-regular expanders, Combinatorica, 14(4) 1994, 379-416.

[2] N. Alon, Eigenvalues and expanders, Combinatorica 6(1986), 83-96.

[3] N. Alon and V. D. Milman, Eigenvalues, expanders and superconcentrators, Proc. 25^{th} Annual Symp. on Foundations of Computer Science (FOCS), Singer Island, Florida, IEEE (1984), 320-322. Also: λ_1, isoperimetric inequalities for graphs and superconcentrators, J. Combinatorial Theory, Ser. B 38(1985), 73-88.

[4] N. Alon and Y. Roichman, Random cayley graphs and expanders, Random Structures and Algorithms 5 (1994), 271-284.

[5] Y. Bilu and N. Linial, Lifts, discrepancy and nearly optimal spectral gaps, Combinatorica, to appear.

[6] J. Dodziuk, Difference equations, isoperimetric inequality and transience of certain random walks, Trans. Amer. Math. Soc. 284 (1984), 787–794.

[7] O. Gabber and Z. Galil, Explicit constructions of linear-sized superconcentrators, J. of Computer System Sciences, 22 (3) 1981, 407-420.

[8] M. Gromov, Filling Riemannian manifolds, J. of Differential Geometry, 18(1) (1983), 1-147.

[9] S. Hoory, N. Linial and A. Wigderson, Expander graphs and their applications, Bulletin of the AMS, Vol 43 (4), 2006, 439-561.

[10] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan graphs, Combinatorica 8(3) (1988), 261-277.

[11] G. Margulis, Explicit constructions of expanders, Problemy Peredacl Informacii, 9 (1973), 71-80.

[12] G. A. Margulis, Explicit group-theoretical constructions of combinatorial schemes and their application to the design of expanders and superconcentrators, Problems of Information Transmission 24(1988), 39-46.

[13] M. Pinsker, On the complexity of a concentrator, 7^{th} annual teletraffic conference (1973), 1-4.

[14] O. Reingold, S. Vadhan and A. Wigderson, Entropy waves, the zig-zag graph product, and new constant-degree expanders, Annals of Mathematics, 155 (1) (2002), 157-187.

Appendix: Proof of right inequality of Theorem 1: Let $Q = dI - A$ be the Laplace matrix of G. Our goal is to prove that all but one of the eigenvalues of Q are at least $\frac{1}{2}\delta^2 d$. Let $z = (z_1, z_2, \ldots, z_n)$ be an eigenvector of Q with the smallest nontrivial eigenvalue λ, where

$V(G) = \{1, 2, \ldots, n\}$. Recall that for every set U of at most half the vertices of G there are at least $c|U|$ edges between U and its complement, where $c = \delta d$ is some positive constant. Clearly $\sum_i z_i = 0$. Without loss of generality assume that $m \leq n/2$ of the entries of z are positive (otherwise, replace z by $-z$), and that $z_1 \geq z_2 \geq \ldots \geq z_m > 0 \geq z_{m+1} \geq \ldots \geq z_n$. Define $x_i = z_i$ for $i \leq m$, and $x_i = 0$ otherwise. Since $x_j = 0$ for all $j \geq n/2$,

$$\sum_{ij \in E} |x_i^2 - x_j^2| = \sum_{ij \in E, i<j} (x_i^2 - x_j^2) \geq \sum_{i: i<n/2} (x_i^2 - x_{i+1}^2) ci = c \sum_{i=1}^{n} x_i^2. \tag{2}$$

Note that $(Qz)_i = \lambda z_i$ for all i and hence $\lambda = \frac{\sum_{i=1}^m (Qz)_i z_i}{\sum_{i=1}^m z_i^2}$. However,

$$\sum_{i=1}^{m} (Qz)_i z_i = \sum_{i=1}^{m} (d z_i^2 - \sum_{j, ij \in E} z_i z_j) = \sum_{i,j \leq m, ij \in E} (z_i - z_j)^2 + \sum_{i \leq m, j > m, ij \in E} z_i (z_i - z_j) \geq \sum_{ij \in E} (x_i - x_j)^2.$$

As $\sum_{i=1}^m z_i^2 = \sum_{i=1}^n x_i^2$ we conclude, using Cauchy Schwartz (twice) that

$$\lambda \geq \frac{\sum_{ij \in E} (x_i - x_j)^2}{\sum_{i=1}^n x_i^2} = \frac{\sum_{ij \in E} (x_i - x_j)^2 \sum_{ij \in E} (x_i + x_j)^2}{\sum_i x_i^2 \sum_{ij \in E} (x_i + x_j)^2} \geq \frac{(\sum_{ij \in E} |x_i^2 - x_j^2|)^2}{\sum_i x_i^2 2d \sum_i x_i^2} \geq \frac{c^2}{2d},$$

where the last inequality follows from (2). Therefore, $\lambda \geq \frac{c^2}{2d} = \frac{1}{2} \delta^2 d$. ■