1 Introduction

Until now we talked about equilibrium models in mechanisms with partial information. This means without any assumptions of probability on the input. Now, each of the participants is familiar with his private information, but not the others’. However, he can derive the expected value of the others’ private information knowing their distribution.

2 Bayesian-Nash Equilibrium

Each player has its type, $T_i$, and we wish to implement the social choice function, which is defined as:

$$f : T_1 \times \ldots \times T_n \rightarrow A$$

$S_i$ is the collection of strategies (space of actions) for player $i$. We assume that each player knows the distribution $D_i$ of the $T_i$ of any other player. In addition we define the following functions:

Given the actions of all players, $a$ is the result function (the chosen alternative).

$$a : S_1 \times \ldots \times S_n \rightarrow A$$

Given the actions of all players, $p_i$ is the payment of player $i$.

$$p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$$

So, what strategy should player $i$ have, for each possible $t_i$, assuming he knows the distribution for every other player’s $T_j$? $S_i(t_i)$ should maximize the following expectancy:

$$E_{t_{-i} \sim D_{-i}}[v_i(t_i, a(s_i, s_{-i}(t_{-i})))) - p_i(s_i, s_{-i}(t_{-i}))]$$

Definition 1 Functions $s_1() \ldots s_n()$ are said to be in a Bayesian-Nash Equilibrium if every $s_i$ is maximizing the defined expectancy.

Note: In many cases there is no Bayesian-Nash equilibrium at all between the strategies.
3 Analysis of Bayesian-Nash Equilibrium in First-Price Auction

Let’s assume for the analysis that we have 2 players, Alice and Bob, and the private information of each is uniformly distributed in $[0,1]$. The private information here represents the value of the product to the player. Let $t_{Alice}=a$, and $t_{Bob}=b$. Every player will announce a value smaller than his real value. Therefore, in order to discover how much smaller the announced value is, we’ll go through the following analysis.

Let $X$ denote the announced value of Alice. ($X$ is a function of $a$). Let $Y$ denote the announced value of Bob. ($Y$ is a function of $b$).

Let $U_i$ be the utility function of player $i$. $U_i= v_i(t_i, a(s_i, s_{-i}(t_{-i}))) - p_i(s_i, s_{-i}(t_{-i}))$

$U_A(x, y) = \begin{cases} 
  a - x & x > y \\
  0 & \text{otherwise.}
\end{cases}$

How do we find the Bayesian-Nash Equilibrium? Assume $y=b$ (Bob is announcing his real value). What is Alice’s best reply strategy in that case? She should find the value of $x$, which maximizes the following expectancy: $E_bU_A(x, y)=E_bU_A(x, b)$ Let’s assume that $x$ is a constant, then this expectancy will equal to: $Pr(b<x)[a-x]=x(a-x)$ What value of $x$ will maximize this expectancy? By simple calculations, we come to the conclusion that the value should be $x = \frac{a}{2}$

Due to the fact that Alice and Bob are symmetric players, it is clear that if $x=a$, Bob’s best reply strategy will be to announce half of his real value for the product: $y = \frac{b}{2}$

Now, let’s assume $y=b/2$. What will be Alice’s best reply strategy in this case? She should find the value of $x$, which maximizes the following expectancy of her profit: $E_bU_A(x, y)=E_bU_A(x, b/2)=Pr((b/2)<x)[a-x]=$

$= \begin{cases} 
  a - x & x \geq 1/2 \\
  2x(a - x) & \text{otherwise.}
\end{cases}$

What is the value of $x$ which maximizes this expectancy? We’ll note that this $x$ must be in the range $x<1/2$. Otherwise it is true that:

$a - x \leq 2x(a - x)$ (1)

Due to equation (1) it is clear that the value which maximizes the expectancy is in the range $x<1/2$ (or $x=1/2$). So, by derivation of $2x(a-x)$ we immediately see that the maximum (which is therefore global) is obtained when $x = \frac{a}{2}$

Due to the fact that Alice and Bob are symmetric players, it is clear that if \( x = a/2 \), Bob’s best reply strategy will be to announce half of his real value for the product:

\[
y = \frac{b}{2}
\]

**Conclusion 2** *The Bayesian-Nash Equilibrium was obtained when each player’s strategy is to announce half of his real value.*

Note: There are no more symmetric Bayesian-Nash Equilibriums here.

### 4 From the Seller’s Point of View

When is the seller’s profits expectancy greater - in first or second price auction? At the Bayesian-Nash Equilibrium, this expectancy equals to:

\[
E_{a,b} \max \left( \frac{a}{2}, \frac{b}{2} \right) =
\]

\[
= \frac{E_{a,b} \max(a, b)}{2} =
\]

\[
= \int_a \int_b \max(a, b) \frac{1}{2} =
\]

\[
= \frac{\int_a a + (1-a)^2}{2} =
\]

\[
= \frac{\int_a 1 + a^2}{2} =
\]

\[
= \frac{1}{2} \int_a 1 + a^2 =
\]

\[
= \frac{1}{2} + \frac{1}{2} \int_a a^2 = \frac{1}{3}
\]

Note that in the case of a second-price auction we have the following equation:

\[
E_{a,b} (\min(a, b)) = \frac{1}{3}
\]

Therefore, the expectancy of the seller’s profit is equal in both a first and a second price auctions.
5 The Revenue Equivalence Theorem

Definition 3 The following function $f$ is implemented by some mechanism:

$$f : T_1 \times ... \times T_n \rightarrow A$$

$p_i$ is the payment of player $i$.

$$w_i(t_i) = \text{Pr}_{t_{-i}}[i \text{ wins if his value equals to } t_i]$$

$$p_{t_{-i}} = \mathbb{E}_{t_{-i}}[p_i(t_i, t_{-i})]$$

The group of alternatives can be divided into 2 parts: those alternatives which satisfy the player, and those that don’t. For each player $i$ we denote $w_i$ to be the winning set of player $i$ - the set of alternatives, which satisfy him.

$$v_i(t_i, a) = \begin{cases} t_i & a \in w_i \\ 0 & \text{otherwise.} \end{cases}$$

According to the definitions above, we have:

$$w_i(t_i) = \text{Pr}_{t_{-i}}(f(t_i, t_{-i}) \in w_i)$$

$$p(t_i) = \mathbb{E}_{t_{-i}}(p_i(t_i, t_{-i}))$$

Note: Whenever the partition is known, one can derive exactly how much was paid.

Claim 4 $W$ determines $p$ in a one-to-one relation. We’ll assume that $p(0) = 0$ (the payment of a player whose real value is 0, will be 0). Two mechanisms, which implement the same social choice function $f$, have an equal expectancy of the players’ payments, and therefore, same expectancy of the seller’s profit.

Proof: A player whose real value is $t_i$, will not profit from announcing a different value $\tilde{t}_i$. What is the player’s profit if instead of announcing his real value $t_i$ he announces $\tilde{t}_i$?

$$\mathbb{E}(\text{player } i's \text{ profit when he announces } \tilde{t}_i \text{ and his value is } t_i) = w(\tilde{t}_i) \cdot t_i - p(\tilde{t}_i)$$

$$w(\tilde{t}_i) \cdot t_i - p(\tilde{t}_i) \leq w(t_i) \cdot t_i - p(t_i)$$

$$t_i(w(\tilde{t}_i) - w(t_i)) \leq p(\tilde{t}_i) - p(t_i)$$

If a player whose real value is $t_i$ announces $t_i$, it is true that:

$$\tilde{t}_i(w(t_i) - w(\tilde{t}_i)) \leq p(t_i) - p(\tilde{t}_i) \quad / \cdot (-1)$$

$$\tilde{t}_i(w(\tilde{t}_i) - w(t_i)) \geq p(\tilde{t}_i) - p(t_i)$$

$$t_i(w(\tilde{t}_i) - w(t_i)) \leq p(\tilde{t}_i) - p(t_i) \leq \tilde{t}_i(w(\tilde{t}_i) - w(t_i))$$

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Let's substitute \( \tilde{t}_i = t_i + \epsilon \) and divide by \( \epsilon \)

\[
\frac{t_i (w(t_i + \epsilon) - w(t_i))}{\epsilon} \leq \frac{p(t_i + \epsilon) - p(t_i)}{\epsilon} \leq (t_i + \epsilon) \cdot \frac{w(t_i + \epsilon) - w(t_i)}{\epsilon}
\]

The left and right hand sides of the inequality \( \rightarrow t_i \cdot w'(t_i) \), when \( \epsilon \rightarrow 0 \)

Under the assumptions that \( w \) is continuous and derivative, we proved that \( p \) is derivative, therefore

\[
\frac{p(t_i + \epsilon) - p(t_i)}{\epsilon}
\]

has a limit which equals to \( p'(t_i) \)

\[\blacksquare\]

Conclusion 5

\[
p(\tau_i) = \int_0^{\tau_i} t_i \cdot w(t_i) dt_i
\]