1 A simple Market: supply, demand, valuation, equilibrium, prices, social welfare, and some challenges (part 2)

Consider the following market scenario: suppose we have $N$ buyers and $M$ sellers, such that the $i$'th buyer has a function $v_i(x)$ that specifies the value of $x$ units to $i$, and the $j$'th seller has a function $c_j(y)$ that specifies the cost of $y$ units to $j$, where $x$ and $y$ are the number of units. We call these functions the valuation function of $i$ and the cost function of $j$ respectively.

For simplicity we consider first a simple (degenerate case) model of a market.

In this simple case each valuation function is defined by the pair $(q, p)$ in the following way:

$$v_i(x) = \begin{cases} 
    x \cdot p & \text{if } x \leq q; \\
    q \cdot p & \text{if } x > q.
\end{cases}$$

![figure 1: valuation function](image)

Note that $p$ is the maximum price per unit the buyer is willing to pay and $q$ is the maximum number of units the buyer assigns value for. This makes the buyer indifferent to more then $q$ units of the product. Note also that in this simple scenario, as in most cases along this course, we treat $q$ as an integer.

In a similar way each seller has a cost function $c_i(y)$ that is defined by $(p, q)$:
As can easily be seen the seller’s cost function $c_i(y)$ is linear in $y$ and then goes to infinity at $q$. This means that selling more than he has (or willing to give) is unacceptable (or priceless).

After describing the setting we now turn to see what we would like to achieve:

The Goal:

We would like to supply a list of $x_i$, $y_j$ where $x_i$ is the amount of units that the $i$’th buyer bought and $y_j$ is the amount of units that the $j$’th seller sold, where $\sum_i x_i = \sum_j y_j$ (This means that the amount that exchanged hands is equal), such that the social welfare (gains from trade) i.e:

$\sum_i v_i(x) - \sum_j c_j(y)$ is maximized.

Note that the cost is negative and there for appears in a negative sign.

Before turning to see the behavior of this model lets see what are the limitations we impose on such a model:

1) No externalities.

2) Every player in the market has its own goal and its own interest.(specifically one does not care about the other’s interests)

Definition: pareto optimality

A solution is pareto optimal if there is no other solution that all players prefer (weak) at the same time. That is, it is impossible to improve a certain player’s state without diminishing another’s.
Figure 3: It's obvious that solution b is not pareto optimal because it is possible to improve the state of player z (solution a) without harming that of the other players.

So what is the solution?
As you will prove in exercise 1, a solution that will not maximize (*) is not pareto optimal (and therefore is an unreasonable solution). In other words (remember logics) a pareto optimal solution maximizes (*).

We have already seen an algorithm that solves this problem last week: (see lecture 1) Just as a reminder, remember that the first stage of the algorithm involves in sorting the preferences of the buyers and the sellers separately and then traversing both lists once. This greedy algorithm just goes over the list from bottom up until one of the lists, the buyers' or the sellers' ends.
The complexity of this algorithm is straightforward and is bounded by the sorting operation which takes $O(N \log(N) + M \log(M))$. The greedy part of the algorithm is linear in N and M.

**Theorem 1** The above algorithm achieves pareto-optimality.

**Proof:** (by negation)
At this stage we will only draw the rough lines of the proof, since an elaborate one will be given later in this lecture.
The only possible solution is that all top sellers receive all the q’s and beyond them all other sellers do not receive anything (besides one seller who can sell a partial amount). Otherwise we can switch between the sellers and improve the situation, in contradiction to optimality.

In addition, the point where we left off is optimal since any sale which takes place after the algorithm stops can simply be canceled and thus we can only improve the situation.

So far we have just handled the distribution mechanism. We now turn to discuss the actual money transfer. We would like to know at the end of the process who pays/receives what,
so that everybody is “happy”. To do that we present the concept of an equilibrium state.

The idea of an equilibrium state is that if such a state exists, then no participant will have
incentive to move from its current state while if no such state exist then there will always
be a participant that will have the incentive to move.

**Definition:** Equilibrium price for a distribution

The price $p^*$, is called an equilibrium point for a distribution $x_1, \ldots, x_n, y_1, \ldots, y_m$

if for every buyer

$p_i > p^* \Rightarrow x_i = q_i$

$p_i < p^* \Rightarrow x_i = 0$

and for every seller:

$p_i > p^* \Rightarrow y_i = 0$

$p_i < p^* \Rightarrow y_i = q_i$

In the case where $p^* = p_i$ either one of the possibilities may hold.

**Theorem 2** The first welfare theorem. If $p^*, x_1, \ldots, x_n, y_1, \ldots, y_m$ is an equilibrium then

$\sum_i v_i(x_i) - \sum_j c_j(y_j)$ is maximal.

Before turning to the proof we point the reader to the fact that this very elegant theorem is
in fact a sort of a simple display of the invisible hand that was introduced by Adam Smith.

We also point the reader to the fact that the below proof infact proves that the algorithm we
proposed earlier works.

**Proof:**

Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be a different distribution (that agrees with $\sum_i x_i = \sum_j y_j$) Then

for each and every buyer $i$:

$v_i(x_i) - x_ip^* \geq v_i(x_i^*) - x_i^*p^*$

and for each and every seller $j$:

$y_jp^* - c_j(x_j) \geq y_j^*p^* - c_j(y_j^*)$

If we add all these these inequalities over $j$ and $j$ we get (the inequality holds)

$\sum_i (v_i(x_i) - x_ip^*) + \sum_j (y_jp^* - c_j(x_j)) \geq \sum_i (v_i(x_i^*) - x_i^*p^*) + \sum_j (y_j^*p^* - c_j(y_j^*))$

if we take decompose the summation:

$\sum_i v_i(x_i) - \sum_i x_ip^* + \sum_j y_jp^* - \sum_j c_j(x_j) \geq \sum_i v_i(x_i^*) - \sum_i x_i^*p^* + \sum_j y_j^*p^* - \sum_j c_j(y_j^*)$

We can now take $p^*$ out of the summation and get

$\sum_i v_i(x_i) - \sum_j c_j(x_j) - p^*(\sum_i x_i - \sum_j y_j) \geq \sum_i v_i(x_i^*) - \sum_j c_j(y_j^*) - p^*(\sum_i x_i^* - \sum_j y_j^*)$
And under the assumption stated earlier \( \sum_i x_i = \sum_j y_j \) We get the proof of optimality.

The price in which we stop (i.e. the last satisfied \( p \) in the algorithm) is the equilibrium price. And from the above theorem we have that this price is the optimal price.

**Theorem 3** The second welfare theorem. In a market like ours there exist an equilibrium

**Proof:** By construction: (see the above algorithm).

. What happens when we take a different value function? Remember the naïve value function we defined earlier (see figure 1). How can we make the value function more elaborate: One possibility would be multiple preferences for each buyer. How does this valuation function look like?

<table>
<thead>
<tr>
<th>price</th>
<th>quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

*figure 4: multiple preferences valuation function*

Will the above algorithm still work?
The answer is Yes and No!

**Yes** - When the valuation function is convex (decreasing marginal utility) since we can treat each segment of preference as a different buyer or seller and in such a description of the matters the 'first buyer' will always make the buy before the 'second one' (Remember the buyers are ordered in a decreasing order).

**No** - When the valuation function is concave, (increasing marginal utility)
So, we can generalize the second welfare theorem and say that the algorithm works for any convex valuation function. (decreasing marginal utility)

Let's discuss a new scenario:
All or non case, in this scenario a buyer (seller) either buys (sells) all the amount he wishes or buys (sells) nothing.

Will there be an equilibrium?
Let's look at an example:
Sellers:
one seller sell 2 units in 0.
Buyers: (all or non)
a. 1 units 10 each
b. 2 units 8 each

The optimal solution assigns 2 units for b, but there is no equilibrium price that gives us this solution, because if the equilibrium price will be below 8 then there is a demand for 3 units, whereas only 2 are available. while, if the equilibrium price will equal 8 then the first one will need to buy but he won’t buy (he wants only one unit), etc...
Thus, in the case of all or non the algorithm does not work!

Let's find a new algorithm:
Note that this problem is in fact the knapsack problem, (and therefore the same solution will hold) As an illustration, we will look at a special case where we have one seller that wants to sell $M$ units in price 0, and $N$ buyers that want to buy $x_i$ units in price 1 (each).
We will try to find a subset of $\{x_i\}$ that amounts to $M$. this is an NPC problem. Non polynomial in the input size. But the unary knapsack problem is solvable and the problem

\[ \begin{array}{c}
\text{Buyer Input} \\
20 & 17 \\
\end{array} \]

\[ v_j(20) \]

\[ 20 \]

\textit{figure 5: valuation function of all or non case}
we handle is pseudo polynomial i.e. polynomial in \((N, M, \sum q_i)\) This is great example of dynamic programming.

The algorithm:
We start by building two matrices, a matrix of buyers \(B(i, q)\) \(1 \leq i \leq N\) and a matrix of sellers \(S(i, q)\) Where at the \(B(i,j)\) entry we write the maximal value that the sellers can achieve from selling \(q\) units.

\[
B(0, j) = 0 \\
B(i, q) = \max \left[ B(i - 1, q), B(i - 1, q - q_i) + p_i q_i \right]
\]

Where the \(S(i,j)\) entry is the minimum cost if selling exactly \(q\) units. The update rule of the \(S\) is similar to that of \(B\).

And at the end we will look at the

\[
\max_q \left[ B(n, q) - S(m, q) \right]
\]

Coming up next on  What happens when the participants are dishonest (where the participants can influence the solution by being dishonest) What happens when on try to sell different products with dependencies ?

Some words (and a picture) about Adam Smith:

\(\text{Adam Smith}\)

(baptised June 5, 1723 O.S. / June 16 N.S. July 17, 1790) was a Scottish moral philosopher and a pioneering political economist. One of the key figures of the intellectual movement known as the Scottish Enlightenment, he is known primarily as the author of two treatises:
The Theory of Moral Sentiments (1759), and An Inquiry into the Nature and Causes of the Wealth of Nations (1776). The latter was one of the earliest attempts to systematically study the historical development of industry and commerce in Europe, as well as a sustained attack on the doctrines of mercantilism. Smith’s work helped to create the modern academic discipline of economics and provided one of the best-known intellectual rationales for free trade, capitalism, and libertarianism. Adam Smith is now depicted on the back of the Bank of England 20 note.

And one of his quote from The Wealth of Nations:

“He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it. By preferring the support of domestic to that of foreign industry, he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention.”