Multi-unit Auctions with Budget Limits

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Abstract

We study multi-unit auctions where the bidders have a budget constraint, a situation very common in practice that has received very little attention in the auction theory literature. Our main result is an impossibility: there are no incentive-compatible auctions that always produce a Pareto-optimal allocation. We also obtain some surprising positive results for certain special cases.

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1 Introduction

The starting point of almost all of auction theory is the set of players' *valuations*: how much value (measured in some currency unit) does each of them assign to each possible outcome of the auction[1]. When attempting actual implementations of auctions, a mismatch between theory and practice emerges immediately: *budgets*. Players often have a maximum upper bound on their possible payment to the auction – their budget. This budget limit is not adequately expressible in existing auction theory[2]. As budgets are central elements in most of economic theory (e.g., in the Arrow-Debreu market model), it is quite surprising that so little attention has been paid to them in auction theory, and in particular in auctions of multiple units or goods. The reason seems to be that budgets take us out of the clean "quasi-linear" setting, which changes many of the "rules of the game". Previous work on this issue in economic theory mostly focused on comparing classical auctions formats under budget constraints (e.g., [2]). We are aware of only two recent papers that have attempted explicitly designing auctions for this setting [5, 4].

The Inadequacy of the Quasi-linear Model

Some previous papers (e.g. [6, 7]) have attempted modeling the budget limit as an upper bound on the value obtained by the bidder rather than on his payment. Specifically they model a player with valuation v and budget B as though having a valuation $v' = \min(B, v)$. This model maintains the quasi-linear setting but misses the point when the budget limit is a real constraint, i.e. is lower than what the payment would be without it. A typical case where budgets play a central role is an auction for multiple items (homogenous or heterogenous), where bidders have a value for each unit and a budget limit that constrains the number of units they can obtain, i.e. the budget is significantly smaller than the combined value of all items[3]. This is in contrast to the unrealistic quasi-linear modeling where the constraint is a fixed demand curve such as an upper bound q on the number of units desired by the bidder. In such cases, once a value equalling the budget is reached, the $\min(v, B)$-model incorrectly models the marginal value of additional goods as being zero. This would lead to several artifacts, in particular "VCG" payments will be identically zero, loosing not only all revenue but also incentive-compatibility itself. From an allocation point of view, any allocation in which all bidders' values reach their budgets would be considered efficient (maximize social-welfare) including those that are not even Pareto-efficient. We thus conclude that addressing budgets properly requires transcending the pure quasi-linear setting, as usually done in the rest of economic theory.

Our Model

In this paper we study multi-unit auctions with budget limits. Our model is simple: There are m identical indivisible units for sale, and each bidder i has a private value v_i for each unit, as well as a private budget limit b_i on the total amount he may pay. We also consider the limiting case where m is large by looking at auctions of a single infinitely-divisible good. Our assumption is that bidders are utility-maximizers, where i's utility from acquiring x_i units and paying p_i is $u_i = x_i \cdot v_i - p_i$, as long as the price is within budget, $p_i \leq b_i$, and is negative infinity (infeasible) if $p_i > b_i$[4]. As the revelation principle holds in this setting we concentrate on truthful auction mechanisms. In this paper we concentrate on allocational efficiency, only commenting on revenue aspects,

[1] This "quasi-linear" setting is needed due to the Gibbard-Satterthwaite impossibility theorem for the general setting without money.

[2] The nature of what this budget limit means for the bidders themselves is somewhat of a mystery since it often does not seem to simply reflect the true liquidity constraints of the bidding firm. There seems to be some risk control element to it, some purely administrative element to it, some bounded-rationality element to it, and more. What can not be disputed is that these budget constraints are very real to the bidders, in fact, usually more concrete and real than the rather abstract notion of the valuation.

[3] The original motivation for this work, as well as for [5, 4] came from ad-auctions which are certainly such a case. Auctions for spectrum licences and for pollution permits are natural applications as well.

[4] This model naturally generalizes to combinatorial auctions: bidders have a valuation $v_i(\cdot)$ and a budget b_i, and their utility from acquiring a set S_i of items and paying p_i for them is $v_i(S) - p_i$ as long as $p_i \leq b_i$ and negative infinity if the budget has been exceeded $p_i > b_i$. It is interesting to note that the "demand-oracle model" (see e.g. [3]) represents such bidders as well. Analyzing combinatorial auctions with budget limits, even in simple setting such as additive valuations is clearly a direction for future research.

and leaving revenue maximization as a future research direction, already initiated in [4]. As our setting is not quasi-linear, allocational efficiency is not uniquely defined since different allocations are preferred by different players[5]. We thus focus at the weakest efficiency requirement: Pareto-optimality, i.e. allocations where it is impossible to strictly improve the utility of some players without hurting those of others. This strengthens our negative results, while the quality of our positive results may be evaluated by other criteria favored by the reader.

Impossibility Result

Our main result in this paper is a spoiler: there are no incentive-compatible auctions that always produce a Pareto-optimal result.

Main Theorem: There is no Incentive-Compatible Pareto-Optimal auction for any finite number $m > 1$ of units of an indivisible good with any $n \geq 2$ number of players. There is no Anonymous Incentive-Compatible Pareto-Optimal auction for an infinitely-divisible good with any $n \geq 2$ number of players.

This theorem assumes "individual rationality" and "no positive transfers", i.e. that bidders are not paid by the auction nor do they pay more than their value or budget. Without this, the budget limits can be easily side-stepped, e.g., by using a VCG mechanism that pays losers the total value of the others. The anonymity condition, in the infinitely divisible case, means that the auction is symmetric with respect to the bidders, and is likely just a current limitation of our proof. This theorem certainly suggests a research direction of studying approximation in this setting (as already taken in [4] in the context of revenue maximization).

We have looked into special cases for which incentive-compatible Pareto-optimal auctions can be found. Our characterizations for these cases are not only interesting in themselves but also shed light on the type of effects of budgets and serve as intermediate steps for the impossibility theorem above.

Warmup: Competitive Equilibrium and Ascending Auctions

We start by looking at the "competitive equilibrium" (ignoring incentives at first), that is always Pareto-efficient, and for simplicity let's do this in the model of a single infinitely divisible good. A competitive equilibrium consists of a price p and allocation $x_1 ... x_n$ at which each bidder gets his "demand" $x_i = d_i$, where the demand of bidder i at price p is $d_i = 0$ if $v_i < p$, and $d_i = \min(1, b_i/p)$ for $v_i > p$ (For the border case $v_i = p$ any value $d_i \in [0, \min(1, b_i/p)]$ is allowed.)

It is useful to imagine a "continuous-time" ascending auction which can reach this equilibrium: the price starts at $p = 0$ and slowly increases; at any time each bidder declares his demand d_i, and the price continues increasing as long as there is over-demand $\sum_i d_i > 1$. Notice that the demands decrease as p increases so at a certain point $\sum_i d_i = 1$ (notice that the discontinuities in d_i are only when $p = v_i$, at which point our definition allowing any value $d_i \in [0, \min(1, b_i/p)]$ exactly suffices for equality to hold.)

Let us look at this competitive equilibrium in two extreme cases, the first where only the v_i's "matter" ($v_i << b_i$) and the second where only the b_i's matter ($b_i << v_i$). In the first case, as the price increases, bidders drop one after another, changing their demand from 1 to 0. The auction terminates when all bidders drop except for the highest one which happens when the 2nd highest bidder drops. This happens at the 2nd price, which is the lowest equilibrium price. This is the classic English-Auction implementation of the second-price auction and is thus incentive-compatible.

Now let us consider the other extreme case, where $b_i << v_i$. This case is actually quite important as it models the situation where there are many units for sale and no bidder is able to acquire more than a fraction of the total number of units. In this case one can verify that as the price increases, the demands decrease continuously, until equilibrium is reached at price $p = \sum_j b_j$, at which each bidder i gets his *proportional share*: $x_i = d_i = b_i / \sum_j b_j$. This allocation was suggested in [5] who showed that if bidders only aim to maximize their share x_i (rather than their utility) then this is incentive-compatible. It is not difficult to see that for the usual utility-maximizing bidders, this is not incentive compatible: if a bidder is a near-monopsony $b_i >> \sum_{j \neq i} b_j$,

[5] In quasi-linear settings any Pareto-optimal allocation optimizes "social-welfare" – sum of bidders valuations – and thus efficiency is justifiably interpreted as maximizing social-welfare.

then his allocation will be very close to 1, and decreasing his declared budget will only slightly reduce x_i, while significantly reducing his payment. Our first positive observation is that when values are large enough, incentive compatibility is maintained.

Proposition: Let $\alpha_i = b_i / \sum_j bj$ be the budget share of player i. The proportional share auction with $x_i = b_i / \sum_j b_j$ and $p_i = b_i$ is Pareto Optimal and is Incentive Compatible in the range $v_i \geq \sum_j b_j/(1-\alpha)$ for all i.

Notice that the lower bound on v_i is slightly more severe than $v_i \geq \sum_j b_j$ which states that bidders are willing to pay the equilibrium price. Also notice that this auction has excellent revenue properties: it exhausts all budgets.

Ausubel's Clinching Auction

As we have seen above, the ascending auction with equilibrium prices is incentive compatible in the two extreme cases, either when values are much smaller than budgets or the opposite. We wish to further study the intermediate range. We can gain some intuition from the quasi-linear case: in some imprecise sense, to get incentive compatibility, a bidder should not pay the equilibrum price, but rather what would be the price without him.

This is captured beautifully in terms of an ascending auction by Ausubel [1]: as the price in the ascending auction increases, bidders keep decreasing their demands; whenever the combined demand of the other bidders decreases strictly below available supply $\sum_{j \neq i} d_i < m$ then we say that bidder i has "clinched" the remaining quantity at the current price. Thus different amounts of units are acquired by bidders at different prices, and the total payment of a bidder is the sum of the prices of all units that he clinched throughout the auction. Ausubel shows that whenever bidders have downward sloping demand curves, then this auction yields exactly the VCG prices and is thus incentive compatible.

In Ausubel's quasi-linear setting, the demand functions are private information of the bidders and are fixed in advance and thus he gives exact formulas (in terms of these demand functions) for the outcome of the auction. In order to apply this auction type in our setting we view it algorithmically: as bidders clinch units, their demand functions change, taking into account their expenditure so far. I.e., when a unit is clinched by bidder i at price p, bidder i subtracts p from his remaining budget, and at each point during the auction, bidder i's demand is calculated by dividing his remaining budget by the current price. This "adaptive Ausubel's auction" gives an algorithm (or in the infinitely-divisible good case, a continuous time process) specifying the allocation. While this auction is not incentive compatible in general, we show that it is so when budgets are *publically known*. Note that even in this case, the situation is non quasi-linear.

Theorem: For every finite number of units m and any number of players n, the adaptive clinching auction is Pareto-optimal and is incentive compatible when budgets are public information.

Moreover, this auction is unique, at least for the case of two bidders:

Theorem: For the case of two players, with arbitrary publically known budgets b_1, b_2 and for any finite number of units $m \geq 1$, the adaptive clinching auction is the *only* Pareto-optimal and incentive compatible auction.

Exact Analysis

It is not generally easy to analyze the allocation produced by the adaptive clinching auction, especially in its continuous analog, but we present exact forms for several special cases with two players. These were certainly surprising for us, as they do not seem to resemble any previously considered auction format. In all cases, once the exact form is found, it is a straight forward exercise to verify Incentive Compatibility and Pareto-optimality.

1. **Equal Budgets:** This is the case where budgets are equal. Wlog, $b_1 = b_2 = 1$ and $v_1 < v_2$. Here is an incentive compatible Pareto optimal auction: If $v_1 \leq 1$ then the high player gets everything at the second price: $x_2 = 1, p_2 = v_1$ (and $x_1 = 0, p_1 = 0$). Otherwise, the low player gets $x_1 = 1/2 - 1/(2v_1^2)$ and pays $p_1 = 1 - 1/v_1$ and the high player gets $x_2 = 1/2 + 1/(2v_1^2)$ and pays $p_2 = 1$. (In case $v_1 = v_2$, the average allocation is taken for each.)

2. **One Bidder with no Budget Limit:** This is the case where only one of the players is budget-limited. Wlog, $b_1 = 1, b_2 = \infty$. Here is the incentive compatible Pareto-optimal auction: If $\min(v_1, v_2) \leq 1$ then the high player gets everything at the second price: $x_i = 1, p_i = v_j$ (and $x_j = 0, p_j = 0$), where $v_i > v_j$. Otherwise, if $v_2 > v_1$ then the high non-budget-limited player gets everything $x_2 = 1$ and pays $p_2 = 1 + \ln v_1$. If $v_1 > v_2$ then the high player gets $x_1 = 1/v_2$ and pays $p_1 = 1$, while the non-budget-limited player gets $x_2 = 1 - 1/v_2$ and pays $p_2 = \ln v_2$.

We are able to prove uniqueness at least in the first case at least for annonymous auctions:

Theorem: The auction presented above for the case of equal budgets is the only anonymous auction for the infinitely-divisible case that is incentive-compatible and Pareto-optimal.

Future Work

We believe that we have only scratched the surface of analyzing auctions with budgets. We have already mentioned challenges regarding approximations, revenue maximization, and combinatorial auctions.

2 Preliminaries and Notation

2.1 Allocations

We will be considering auctions of a m identical indivisible items as well as the limiting case of a single infinitely divisible good.

We have n bidders, where each bidder i has a value v_i for each unit he gets, and has a budget limit b_i on his payment. Rather than explicitly declaring a bidder's utility of going over-budget to be negative infinity, we will equivalently directly declare such cases to be infeasible.

Definition 2.1 *An allocation is a vector of quantities $x_1 ... x_n$ and a vector of payments $p_1 ... p_n$ with the following properties:*

1. *In the case of finite m, x_i must be a non-negative integer and $\sum_i x_i \leq m$ (*Feasibility*).*
2. *In the case of an infinitely divisible good, x_i must be non-negative real and $\sum_i x_i \leq 1$ (*Feasibility*).*
3. *$p_i \geq 0$ (*No positive transfers*).*
4. *$p_i \leq x_i \cdot v_i$ (*Individual rationality*).*
5. *$p_i \leq b_i$ (*Budget limit*).*

The utility of bidder i from winning x_i quantity and paying p_i is $u_i = x_i \cdot v_i - p_i$.

2.2 Auctions and Incentives

We will be formally considering only direct revelation auctions where bidders submit their value and budget to the auction, that based on this input $v_1 ... v_n$ and $b_1 ... b_n$ calculates the allocation $x_1 ... x_n$ and $p_1 .. p_n$. Our auctions have a very natural interpretation as dynamic ascending auctions, an interpretation that maintains incentive compatibility[6], but for simplicity we will just consider the auction mechanism as a black-box direct-revelation one.

[6]As usual, the incentive compatibility of the iterative versions is only in the ex-post-Nash sense.

Definition 2.2 *A Mechanism is incentive compatible (in dominant strategies) if for every $v_1...v_n$ and $b_1...b_n$ and every possible manipulation v'_i and b'_i, we have that $u_i = x_i \cdot v_i - p_i \geq x'_i \cdot v_i - p'_i = u'_i$, where (x_i, p_i) are the allocation and payment of i for input (v_i, b_i) and (x'_i, p'_i) are the allocation and payment of i for input (v'_i, b'_i).*

A mechanism is incentive compatible for the case of publically known budgets if the definition above holds for all v'_i and the fixed $b'_i = b_i$.

As defined above, we always assume in this paper that auctions are individually rational and have no positive transfers. This in particular implies (and is essentially equivalent to) $x_i = 0 \rightarrow p_i = 0$.

2.3 Pareto-optimality

Definition 2.3 *An allocation $\{(x_i, p_i)\}$ is Pareto-optimal if for no other allocation $\{(x'_i, p'_i)\}$ are all players better off, $x'_i v_i - p'_i \geq x_i v_i - p_i$, including the auctioneer $\sum_i p'_i \geq \sum_i p_i$, with at least one of the inequalities strict.*

Proposition 2.4 *An allocation $\{(x_i, p_i)\}$ is Pareto-optimal in the infinitely divisible case if and only if (a) $\sum_i x_i = 1$, i.e. the good is completely sold, and (b) for all i such that $x_i > 0$ we have that for all j with $v_j > v_i$, $p_j = b_j$. I.e. a player may get a non-zero allocation only if all higher value players have exhausted their budget.*

Proposition 2.5 *An allocation $\{(x_i, p_i)\}$ is Pareto-optimal in the case of finite m if and only if (a) $\sum_i x_i = m$, i.e., all the units are sold, and (b) for all i such that $x_i > 0$ we have that for all j with $v_j > v_i$, $p_j > b_j - v_i$. I.e. a player may get a non-zero allocation only if there is no player with higher value that has larger remaining budget.*

3 The Proportional Share Auction

Definition 3.1 *The proportional share auction for an infinitely divisible good allocates to each bidder i a fraction $x_i = b_i / \sum_j b_j$ of the good and charges him his total budget $p_i = b_i$.*

Proposition 3.2 *Let $\alpha_i = b_i / \sum_j bj$ be the budget share of player i. The proportional-share auction with $x_i = b_i / \sum_j b_j$ and $p_i = b_i$ is Pareto Optimal and is Incentive Compatible in the range $v_i \geq \sum_j b_j / (1 - \alpha)$ for all i.*

Once the auction rule is specified the proof is routine and appears in the appendix.

4 The Adaptive Clinching Auction

We now describe the adaptive clinching ascending auction, and show that it satisfies pareto optimality (PO), individual rationality (IR), and incentive compatibility (IC), when the budgets are known. In the next section we show that it is in fact the *unique* such auction (for two players and any number of items), which enables us to then conclude that when the budgets are private no such auction exists.

The auction keeps for every player i the current number of items she gets q_i, the current total price for these items p_i, and her remaining total budget $b_i = B_i - p_i$. The auction also keeps the global price p and the global remaining number of items q. The price p gradually ascends as long as the total demand is strictly larger than the total supply, where the demand of player i is defined by:

$$D_i(p) = \begin{cases} \lfloor \frac{b_i}{p} \rfloor & v_i > p \\ 0 & \text{otherwise.} \end{cases}$$

As explained in the introduction, if we were to keep the price ascending until total demand would be smaller or equal to the number items, and only then allocate all items according to the demands, then a player could sometimes gain by performing a "demand reduction", thus harming incentive compatibility. Instead, following Ausubel's method, we allocate items to player i as soon as the total demand of the *other* players decreases strictly below the number of currently available items, q. In particular, if at some price p we have $x = q - \sum_{j \neq i} D_j(p) > 0$ then we allocate x items to player i for a unit price p. (This means that the relevant variables are updated as follows: $q_i \leftarrow q_i + x$, $p_i \leftarrow p_i + p \cdot x$, $b_i \leftarrow b_i - p \cdot x$, $q \leftarrow q - x$.) This modification brings back incentive compatibility. The global picture of such an auction is:

The Adaptive Clinching Auction (preliminary version):

1. Initialize all variables appropriately.
2. While $\sum_i D_i(p) > q$,
   (a) If there exists a player i such that $D_{-i}(p) = \sum_{j \neq i} D_j(p) < q$ then allocate $q - D_{-i}(p)$ items to player i for a unit price p. Update all running variables, and repeat.
   (b) Otherwise increase the price p, recompute the demands, and repeat.
3. Otherwise (hopefully $\sum_i D_i(p) = q$): allocate to each player her demand, at a unit-price p, and terminate.

Note that step 2a does not change the amount of over demand, since both the total demand and the total supply are reduced by the same quantity (the number of items that player i gets). Therefore the only factor that affects the over demand is the price; as the price ascends the total over demand decreases. Thus, one would hope that when we reach step 3 we would indeed get $\sum_i D_i(p) = q$, which will enable us to allocate all items at the end (a necessary condition for achieving pareto optimality). However unfortunately this is not quite the case, because the demand functions are not continuous. The demand drops integrally, by definition, and may drop by several items at once. In particular, there are two potentially problemtaic change points: when the price reaches the value v_i, and when the price reaches the remaining budget b_i. The latter point is identified by using:

$$D_i^+(p) = \lim_{x \to p^+} D_i(x),$$

as, for $p = b_i < v_i$, we have $D_i(p) > 0$ and $D_i^+(p) = 0$. Similarly, the former point is identified by using:

$$D_i^-(p) = \lim_{x \to p^-} D_i(x),$$

as, for $p = v_i \leq b_i$, we have $D_i^-(p) > 0$ and $D_i(p) = 0$. We modify the above definition of the auction to use these more refined conditions; (1) the over demand is computed using $D_i^+(p)$, since this ensures that we do not terminate with a price that is just a bit higher than the remaining budget of a player to whom we wish to allocate one last item, and (2) just before termination, if we are left with some non-allocated items, then this must have happened because the final price reached the value of some players (for such a player i we have $D_i^-(p) > 0$ and $D_i(p) = 0$), which caused an abrupt decrease in her demand. These players are indifferent between receiving or not receiving an item, and so we can allocate to them all remaining items.

The Adaptive Clinching Auction (complete version):

1. Initialize all variables appropriately.
2. While $\sum_i D_i^+(p) > q$,
   (a) If there exists a player i such that $D_{-i}^+(p) = \sum_{j \neq i} D_j^+(p) < q$ then allocate $q - D_{-i}^+(p)$ items to player i for a unit price p. Update all running variables, and repeat.

(b) Otherwise increase the price p, recompute the demands, and repeat.

3. Otherwise ($\sum_i D_i^-(p) \geq q \geq \sum_i D_i^+(p)$):

   (a) For every player i with $D_i^+(p) > 0$, allocate $D_i^+(p)$ units to player i for a unit-price p and update the running variables.

   (b) While $q > 0$ and there exists a player i with $D_i(p) > 0$, allocate $D_i(p)$ units to player i, for a unit-price p, and update the running variables.

   (c) While $q > 0$ and there exists a player i with $D_i^-(p) > 0$, allocate $D_i^-(p)$ units to player i, for a unit-price p.

   (d) Terminate.

Let us consider a short example to illustrate the above process. Suppose three items and three players with $v_1 = \infty, B_1 = 1, v_2 = \infty, B_2 = 1.9, v_3 = 1, B_3 = 1$. When the price is below 0.5, each player demands at least two items, and so, for every player, the other players demand more than three items. Therefore no allocations will take place, and the price will keep ascending. At $p = 0.5$, $D_1^+(0.5) = D_3^+(0.5) = 1$ (note that $D_1(0.5)$ and $D_3(0.5)$ are still 2). Thus, player 2 "clinches" one item for a price 0.5. Immediately after that, the demand of player 2 is updated to be 2. The available number of items is 2, and so no player can get any items. At a price 0.7 the demand of player 2 reduces to 1, but this still does not enable the auction to allocate any item to any player. The price keeps ascending until $p = 1$. At this point, $D_1^+(1) = 0, D_2^+(1) = 1, D_3^+(1) = 0$, and so the total demand reduces to be strictly below the number of available items (which is still 2). Thus we enter step 4. In 4a, player 2 gets one item, and in 4b, player 1 gets one item. Note that we do not allocate any item to player 3, though $D_3^-(1) = 1$. Indeed, moving an item from 2 to 3, for example, will violate the pareto optimality. It is not hard to verify that the final result does satisfy PO and IR.

We wish to prove that, in general, for known budgets, this auction algorithm satisfies the three desired properties IR + IC + PO. We also need to show a more fundamental property (all proofs are in the appendix):

Lemma 4.1 *The adaptive clinching auction always allocates all items.*

Lemma 4.2 *The adaptive clinching auction satisfies Individual Rationality (IR) and Incentive Compatibility (IC). Specifically, every truthful player obtains a non-negative utility, and cannot increase her utility by declaring any value different than her true value.*

Lemma 4.3 *The adaptive clinching auction satisfies pareto optimality (PO).*

5 Uniqueness of the Adaptive Clinching Auction

In this section we show that the ascending clinching mechanism is in fact the only mechanism that is truthful, individually rational, and pareto optimal for the setting of publically known budgets. In the next section we utilize this result to show that there is no mechanism if the budgets are private.

Strictly speaking, we do not prove uniqueness for all possible budgets b_1 and b_2, but for "almost" all budgets. This is in a sense the best we can hope for, as, for example, for one item and $b_1 = b_2$ there are indeed multiple possible auctions (which are identical up to tie breaking). The following technical definition attempts to deal with this issue.

Let $S = (S_1, S_2)$ be a partition of $\{1, \ldots, m\}$. Given $b_1, b_2 \geq 0$, define $b_i^{k,S}$ recursively, for each $1 \leq k \leq m$: for $k = m$, $b_1^{m,S} = b_1, b_2^{m,S} = b_2$. For each $1 \leq k \leq m - 1$, if $k \in S_1$ then: $b_1^{k,S} = b_1^{k+1,S}$, $b_2^{k+1,S} = b_2^{k+1,S} - \frac{b_1^{k+1,S}}{k+1}$. if $k \in S_2$ then: $b_1^{k,S} = b_1^{k+1,S} - \frac{b_2^{k+1,S}}{k+1}$, $b_2^{k+1,S} = b_2^{k+1,S}$. We say that b_1 and b_2 are *S-cross free* if for each $1 \leq k \leq m$ we have that $b_1^k \neq b_2^k$. We say that b_1 and b_2 are *cross free* if they are S-cross free for all S.

Notice that given any b_1 and b_2, a small perturbation will make them cross free.

Theorem 5.1 *There is only one truthful, pareto optimal, and individually rational mechanism for m items and 2 players with known budgets b_1 and b_2 that are cross free.*

We present the overview of the proof here, and postpone to the appendix all proofs of lemmas. Without loss of generality we assume that $b_1 < b_2$. For simplicity, throughout the proof we assume that $v_1 \neq v_2$[7]. The proof is by induction on the number of items m. We start with the base case where $m = 1$.

Lemma 5.2 *There is only one truthful, pareto optimal, and individually rational mechanism for one item and 2 players with known budgets.*

We now continue the induction, assuming uniqueness for $m-1$ items, and proving uniqueness for m items. The logic is as follows. We start with some mechanism A for m items that is truthful, pareto optimal, and individually rational. We then explicitly describe the output (and payments) of A on all inputs, except for inputs of the form $v_1, v_2 \geq \frac{b_1}{m}$. To characterize what A does in this range, we use A to construct a new mechanism A_{m-1} for $m-1$ items *and different budgets*. At the beginning A_{m-1} will only be defined on $v_1, v_2 \geq \frac{b_1}{m}$. We will show that the output of A on inputs where $v_1, v_2 \geq \frac{b_1}{m}$ is defined by the output of A_{m-1}.

Now we would like to finish the proof by claiming that A_{m-1} is unique, by the induction hypothesis. However, since A_{m-1} is not defined on all the range of possible valuations, we cannot use the induction hypothesis, as there might be other mechanisms if the range of possible valuations is restricted. To overcome this, we will extend A_{m-1}, and define its output on all valuations in the range. Then we will show that A_{m-1} is pareto optimal, individually rational, and truthful, hence it is unique by the induction hypothesis. Now we can uniquely determine the output of A on all possible valuations, and in particular in the range $v_1, v_2 \geq \frac{b_1}{m}$, as needed.

We start by characterizing the mechanism A for the "easy" case, where $\min(v_1, v_2) \leq \frac{b_1}{m}$ (the proof is similar in spirit to the proof of Lemma 5.2 and is omitted):

Claim 5.3 *Let A be a mechanism for m items that is pareto optimal, individually rational, and truthful. If $\min(v_1, v_2) \leq \frac{b_1}{m}$, then A allocates all items to the player with the highest value i, and i pays $m \cdot v_j$, where j is the other player.*

Fix v_i. We say that t is in the *range* of player $j \neq i$ given v_i if there exists some declaration v_j such that the mechanism allocates t items to player j. The next easy claim proves that at least one $t > 0$ always belongs to the range (see the appendix for a proof):

Claim 5.4 *Let A be a mechanism for m items that is pareto optimal, individually rational, and truthful. Let $v_1 > \frac{b_1}{m}$. There is always a bid v_2 of player 2 that makes player 2 win at least one item.*

At the heart of the proof lies the following lemma (proof is in the appendix):

Lemma 5.5 *Let A be a mechanism for m items that is pareto optimal, individually rational, and truthful. Fix $v_1 > \frac{b_1}{m}$. Let $t_2 > 0$ be the minimal number that is in the range of 2. Then, Player 2's payment for taking t_2 items is* exactly $\frac{t_2 \cdot b_1}{m}$.

Let us now define the mechanism A_{m-1}. A_{m-1} works on budgets $b'_1 = b_1$ and $b'_2 = b_2 - \frac{b_1}{m}$ (notice that b'_1 and b'_2 are cross free). Notice that now it is not necessarily true that $b'_1 \leq b'_2$. We start by defining A_{m-1} on inputs where $v_1, v_2 > \frac{b_1}{m}$: denote the output of A given inputs v_1 and v_2 by $(\vec{x}, \vec{p})$, where x_i is the amount that i gets, and p_i is what i pays. Let the output of A_{m-1} be (x_1, p_1) for player 1 (i.e., as in A), and for player 2 set the output to $(x_2 - 1, p_2 - \frac{b_1}{m})$. In particular, observe that given the output of A_{m-1} on valuations in this range, we can deduce the output of A on the same valuations.

Let us now extend the definition of A_{m-1} to hold also for valuations where $\min(v_1, v_2) \leq \frac{b_1}{m}$. In this case we allocate all items to the bidder with the highest value, and his payment is $m-1$ times the value of the other player.

[7] In fact, if we allow $v_1 = v_2$ then there are multiple auctions that are possible, due to tie breaking.

Lemma 5.6 *A_{m-1} outputs a feasible allocation, and is pareto optimal, individually rational, and truthful.*

By the induction hypothesis, we have that A_{m-1} is unique. By our discussion, this is enough to prove the uniqueness of A and this concludes the proof of the theorem.

6 An Impossibility Result for Private Budgets

Once the public-budgets case is completely analyzed, the impossibility for the private-budget case follows quite easily.

Theorem 6.1 *There is no truthful, incentive compatible, and pareto optimal mechanism if the budgets are private.*

Proof: We utilize our uniqueness result for 2 players with known budgets. Since we characterized exactly how the mechanism behaves with given budgets, it suffices to show an example where a player is better off declaring a different budget than his real one. Notice that although we present the example for two bidders, the result for more bidders follows by adding more bidders with value and budget of zero.

Suppose that $b_1 = 1, v_1 = \infty, b_2 = 1 + \Sigma_{k=2}^{m} \frac{1}{k} - \delta, v_2 = \infty$, for some small $\delta > 0$. (We might add some small perturbation to make b_1 and b_2 cross free.) For each of the first $m-1$ items our auction will allocate the item to player 2 and charge $\frac{1}{k}$ for the k'th item. Then, player 1's budget is finally bigger than player 2's free budget, so player 1 wins the last item with a payment of $1 - \delta$.

Suppose now that player 1 declares $b_1' = 1 + \epsilon$ instead, for small enough ϵ. The allocation of the auction is the same, but player 2 is charged $\frac{1+\epsilon}{k}$ for the k'th item (for $k > 1$). Thus, when the auction allocates the last item, player 2's free budget is smaller than before: $1 - \delta - \Sigma \frac{\epsilon}{k}$. This is also the payment of player 1. Notice that player 1 is allocated one item, just as when declaring v_1, but his payment is smaller, so he better off declaring b_1' instead of b_1. ∎

7 Exact Characterizations in The Infinitely-Divisible Good Setting

While the adaptive clinching auction may be applied in the infinitely divisible setting by treating it as a continuous time process, it is not totally clear how to analyze it in this setting in general. In this section we analyze explicitly two special cases, and for one of them actually prove uniqueness, from which we derive a general impossibility result for the private-budget case.

In the rest of this section we limit ourselves to the case of two bidders.

7.1 Only one bidder with a budget limit

This is the case where only one of the players is budget-limited. For simplicity of notation we will assume without loss of generality that $b_1 = 1$ and $b_2 = \infty$.

Definition 7.1 (Mechanism A) *:*

- *If $\min(v_1, v_2) \le 1$ then the high player gets everything at the second price: $x_i = 1, p_i = v_j$ (and $x_j = 0, p_j = 0$), where $v_i > v_j$.*
- *Otherwise, if $v_2 \ge v_1$ then the high non-budget-limited player gets everything $x_2 = 1$ and pays $1 + \ln v_1$.*
- *Otherwise, if $v_1 > v_2$ then the high player gets $x_1 = 1/v_2$ and pays $p_1 = 1$, while the non-budget-limited player gets $x_2 = 1 - 1/v_2$ and pays $p_2 = \ln v_2$.*

This mechanism was directly derived by looking at the continuous analog of the adaptive clinching auction: as the price increases between 1 and $\min(v_1, v_2)$, the demand of bidder 1 decreases like $d_1(p) = 1/p$. At that point, bidder 2 clinches $1/p^2 = d(1 - d_1(p))/dp$ units at marginal price p, and thus the total payment of bidder 2 up to that point is obtained by integrating the product. At that point, the larger bidder gets the remaining units at the current price. Once this form was derived, the proof of Pareto-optimality and Incentive-compatibility can be given directly and routinely and is postponed to the appendix..

Proposition 7.2 *Mechanism A is Pareto-optimal and is incentive compatible in the case of publically known budgets.*

7.2 Bidders with equal budgets

This is the case where budgets are equal. For simplicity of notation we assume without loss of generality that $b_1 = b_2 = 1$ and $v_1 \leq v_2$.

Definition 7.3 (Mechanism B)

- *If $v_1 \leq 1$ then the high player gets everything at the second price: $x_2 = 1, p_2 = v_1$ (and $x_1 = 0, p_1 = 0$).*
- *Otherwise, the low player gets $x_1 = 1/2 - 1/(2v_1^2)$ and pays $p_1 = 1 - 1/v_1$ and the high player gets $x_2 = 1/2 + 1/(2v_1^2)$ and pays $p_2 = 1$.*

As defined, this mechanism is not totally symmetric, breaking the tie $v_1 = v_2$ "in favor" of v_2. An anonymous mechanism with the same properties can be obtained by "splitting" in case of a tie:

Definition 7.4 (Mechanism C)

- *If $v_1 = v_2 = v \leq 1$ then $x_1 = x_2 = 1/2$ and $p_1 = p_2 = v/2$.*
- *If $v_1 = v_2 = v > 1$ then $x_1 = x_2 = 1/2$ and $p_1 = p_2 = 1 - 1/(2v)$.*
- *If $v_1 \neq v_2$ then run auction B.*

Proposition 7.5 *Mechanism B is Pareto-optimal and is incentive compatible in the case of publically known budgets. Mechanism C is Anonymous, Pareto-optimal and is incentive compatible in the case of publically known budgets.*

Again, once the definition of the auction was found, the proof is routine and appears in the appendix. What is more difficult is proving the uniqueness of this mechanism (at least among annonymous ones). In fact, we did not derive the exact form in this case by direct analysis of the adaptive clinching auction, but rather by solving the differential equation that will appear in the proof of uniqueness which is postponed to the appendix.

Theorem 7.6 *Mechanism C is the only anonymous mechanism for the divisible good setting that satisfies incentive compatibility (IC) and Pareto-optimality (PO).*

From this theorem, we rather easily deduce (proof also in the appendix) that:

Theorem 7.7 *There exists no anonymous, incentive compatible, Pareto-optimal auction for the divisible good setting, for the case of privately known budgets b_1, b_2.*

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Appendix A: Proof for the Proportional Share Auction

Proof: (of proposition 3.2) Pareto-optimality is trivial from proposition 2.4 since we charge bidders their full budget. We now prove incentive compatibility in the specified range. Since the values v_i do not affect the payment or the allocation, it suffices to show that no manipulation of b_i is profitable. Since we charge each bidder his total declared budget, it is clear that declaring $b'_i > b_i$ will lead to the bidder exceeding his budget. Thus it suffices to prove that no smaller declaration $b'_i < b_i$ is profitable. Let $u(z)$ be the utility obtained by bidder i if he declares a budget of $b'_i = z$. Thus $u(z) = v_i \cdot z/(z + \sum_{j \neq i} b_j) - z$. It suffices to show that u is monotonically increasing with z. To verify this, take the derivative with respect to z: $u'(z) = v_i \sum_{j \neq i} b_i/(\sum_j bj)^2 - 1$. This derivative is non-negative, $u'(z) \geq 0$, as long as $v_i \geq (\sum_j b_j)^2 / \sum_{j \neq i} b_j = \sum_j b_j/(1-\alpha)$, as is specified. ∎

Appendix B: Proofs for the Adaptive Clinching Auction

Proof: (of lemma 4.1) Note that step 3 does not reduce the total demand below q, since if we allocate $\Delta = q - D^+_{-i}(p)$ units to player i then the total demand reduces by Δ and the number of available items also drops by the same quantity, so the strict inequality is kept between the total demand and the total number of available items is kept. Now, suppose that the auction enters step 4 at a price p^*. From the above, we get that for any price $p < p^*$ the auction had $\sum_i D^+_i(p) > q$. Define $D(p) = \sum_i D_i(p)$ and define $D^+(p)$ and $D^-(p)$ similarly. Observe that these three functions are monotone non-increasing, that $D^-(p) = D(p) = D^+(p)$ for any continuity point of $D(p)$, and that for a discontinuity point $D^-(p) > D^+(p)$. Therefore if $D^+(p) > q$ for any $p < p^*$ then $D^-(p^*) \geq q$. Thus step 4 is able to allocate all the remaining q items, as the lemma claims. ∎

Proof: (of lemma 4.2) To verify IR, note that whenever a player gets x item at a unit-price p in steps 3, 4a, and 4b in the auction, it follows that $x \leq D_i(p)$, where the demand is computed with respect to the remaining available budget. The definition of the demand function then implies that the player's value for each item is larger than p, and that the remaining budget is larger than $x \cdot p$. If player i gets an item in step 4c, it follows that

$D_i(p) = 0$ and $D_i^-(p) > 0$. The structure of the demand function implies that this can happen only if $p = v_i$, and in addition the available budget at price p is at least $D_i^-(p)$ times p. Thus in this case the player's additional utility from those items is exactly zero.

To verify IC, observe that declaring a value in this auction is equivalent to deciding on the exact price in which to completely drop from the auction. A player cannot gain by staying in the auction after the price reaches the value, since she will pay more than her value for any items that she will receive at these price levels, and cannot gain by dropping before the price reaches the value, since as explained in the previous paragraph the player does not lose from receiving items at any price $p \leq v_i$. ∎

Proof: (of lemma 4.3) To check the PO condition, fix any two players i and j. We need to verify that, if j received any item at all in the auction, then i's remaining budget in the end of the auction is smaller than j's value, i.e. that $b_i < v_i$, or, alternatively, that $v_j \geq v_i$. Consider the last price p at which player j received an item.

First suppose that p is not the price that ended the auction. In this case (step 3), since j received an item, the auction rules imply that $D_{-j}(p)$ is exactly equal to the number of items left after player j was allocated her items. Since the auction allocates all items, and since it is IR, we get that each player $i \neq j$ received, after price p, exactly her demand $D_i(p)$. In particular, this means that the remaining available budget of i is at most p (otherwise the demand of i at p was higher – she could have afford one more item in a price lower than her value). On the other hand, $v_j > p$, since j demanded items at p, and we are done.

Now suppose that p is the price at which the auction ended. The auction rules imply that if i had $D_i^+(p) > 0$ then she received all this demand, and so by the same argument as above she does not have any remaining budget to buy an item from j. A second case is $D_i^+(p) = 0$ and $D_i(p) > 0$. This implies that the remaining budget of player i at this step is $b_i = p$. If player i received her demand $D_i(p)$ at step 4b then the same argument as above still holds. If not, it must be that player j received her items in step 4a or 4b (but not in 4c, since not all players in 4b were awarded their demand). Thus $D_j(p) > 0$ hence $v_j > p = b_i$ and a pareto improvement cannot take place. The last case is $D_i(p) = 0$ and $D_i^-(p) > 0$. This implies that $p = v_i$, and since $v_j \geq p$ this again rules out the possibility of a pareto improvement. ∎

Appendix C: Proofs for Theorem 5.1

Proof of Lemma 5.2

Proof: We will show that only possible mechanism is the following: the winner is the player i that maximizes $\min(b_i, v_i)$. The winner pays the mechanism $\min(b_j, v_j)$, where j is the other player[8].

It is easy to see that the above mechanism is indeed truthful. We now prove that this is the only possible mechanism. The proof proceeds by considering all possible cases:

- $\min(v_1, v_2) \leq b_1$: if the item is allocated to player i with $v_i < v_j$, then the allocation is not pareto optimal: player j can pay player i $p = \min(v_j - \epsilon, b_i)$, for some small ϵ (notice that $p \leq b_j$, so player j can indeed pay player i this amount), get the item, and all players are better off.

- $v_1, v_2 > b_1$: we claim that player 2 must win the item. First observe that if $v_2 > v_1$ and $v_1 < b_2$, then the only pareto optimal allocation allocates the item to 2 (in the other allocation player 2 can buy the item from 1, and they are both better off). Suppose that there exists some v_2' such that 2 wins the item (notice that his paiement is at most b_1). By truthfulness, any declaration of $b_1 + \epsilon$, for any $\epsilon > 0$ should make him win the auction: else, his profit is zero, but by declaring v_2' he gains some positive profit. However, notice that this allocation is not pareto optimal for some small enough $\epsilon > 0$, by our discussion.

[8] Notice that if the b_1 and b_2 are not cross free, i.e., $b_1 = b_2$, then indeed this auction is not uniquely defined as if $v_1, v_2 > b_1 = b_2$ we can break ties in favor of both players, and still get a valid output

Notice that the resulting output is indeed identical to the mechanism described above. ∎

Proof of Claim 5.4

Suppose there exists some $v_1 > \frac{b_1}{m}$ such that there is no declaration v_2 that makes player 2 win at least one item. That is, player 1 always takes all items. Since the budget of 1 is b_1, the payment for m is at most b_1. Notice that whenever $v_1 > \frac{b_1}{m}$, taking all items is the most profitable alternative for 1, since by Claim 7.9 the marginal payment of taking any additional item is at least $\frac{b_1}{m}$. Finally, observe that by truthfulness for $\frac{b_1}{m} < v_1' < v_2$, 1 must be allocated all items. However this is not pareto optimal, by Proposition 2.5.

Proof of Lemma 5.5

We will prove this gradually. Some of the claims we prove will be useful later.

Claim 7.8 *Let A be a mechanism for m items that is pareto optimal, individually rational, and truthful. Fix $v_j > \frac{b_1}{m}$. Let i be the other player. Suppose there exists some $t_i > 0$ that is the minimal number that is in the range of i given v_j. Then the payment of player i for taking exactly t_i items is at least $\frac{t_i \cdot b_i}{m}$.*

Proof: We already know by Claim 5.3 that if player i value is below $\frac{b_1}{m}$ he gets no items at all. Thus, in order to get some items, he must bid at least $\frac{b_1}{m}$. Thus his threshold value is at least $\frac{b_1}{m}$, and clearly if i gets t_i items, he must pay at least $\frac{t_i \cdot b_i}{m}$. ∎

Claim 7.9 *Let A be a mechanism for m items that is pareto optimal, individually rational, and truthful. Fix $v_j > \frac{b_1}{m}$. Suppose there exists* some $t_i > 0$ *in the range of i given v_j. Then player i has to pay at least $\frac{t_i \cdot b_i}{m}$.*

Proof: Let $t > 0$ be the minimal number in the range of i. By the previous claim we have that the payment for taking i is $\frac{tb_i}{m}$. Assume, towards a contradiction, that there is some (minimal) t_i' in the range of i such that i pays less than $\frac{t' \cdot b_i}{m}$. Let $g < t'$ be the largest predecessor in the range of i. By our assumption, the payment is at least $\frac{g \cdot b_i}{m}$. Now observe that taking t' items is more profitable than taking g items, as the average marginal utility of every additional item is more than the average payment. Hence player i will never select to take g items, in contradiction to our assumption that g is in the range. ∎

We now continue with the proof of of lemma 5.5. By Claim 7.8 we have that the payment of player 2 for t_2 is at least $\frac{t_2 \cdot b_2}{m}$. Let us show that it is not more than this expression. We will show that player 1 cannot take all m items (i.e., m is not in his range), which is equivalent to claiming that player 2 takes at least one item, whenever $v_2 > \frac{b_1}{m}$. Finally, recalling that player 2 gets no items if $v_2 \leq \frac{b_1}{m}$ (because $v_1 > \frac{b_1}{m}$), we have that the threshold value for taking t_2 items is $\frac{b_i}{m}$ and thus the payment of 2 for taking t_2 items is exactly $\frac{t \cdot b_i}{m}$, by standard incentive compatibility arguments.

We now show that player 1 cannot take all items. If $v_2 > \frac{b_1}{m}$, then 1 might win some items only if $v_1 > \frac{b_1}{m}$. Thus his payment for getting the minimal $t' > 0$ items in his range is at least $\frac{t' \cdot b_1}{m}$. By Claim 7.9 his payment from taking all items, if this alternative is in the range, is at least b_1. Since his budget is b_1 the payment is exactly b_1. Hence, every $v_1' > \frac{b_1}{m}$ will make him win all m items, which is not pareto optimal for $v_2 > v_1 > \frac{b_1}{m}$. A contradiction.

Proof of Lemma 5.6

During the proof we abuse notation a bit and identify the output of A with A, and the output of A_{m-1} with A_{m-1}.

Claim 7.10 A_{m-1} *outputs a feasible allocation and is individually rational.*

Proof: If $\min(v_1, b_1) < \frac{b_1}{m}$, then the loser pays nothing by definition. Else, if player 1 is allocated no items in A_{m-1}, then he pays nothing, since A is individually rational and 1 gets nothing also in A. Consider the case where player 2 is allocated no items in A_{m-1}. It means that it was allocated exactly one item in A, and by Lemma 5.5 his payment is $\frac{b_1}{m}$ in A, hence in A_{m-1} his payment is 0. The feasibility of the allocation follows similarly. ∎

Claim 7.11 *A_{m-1} is pareto optimal.*

Proof: Again, we consider several cases. If $v_1, v_2 > \frac{b_1}{m}$ then we observe that each player has the same amount of unused money from his budget: player 1 is allocated and charged the same as in A, and player 2 is charged $\frac{b_1}{m}$ less, but it holds that $b'_2 - b_2 = \frac{b_1}{m}$. Also notice that a player that was allocated no items in A will be allocated no items also in A_{m-1}. Thus, if both players want to exchange an item in A_{m-1}, they both want to exchange it also in A. However, by our assumption A is pareto optimal, so this cannot happen.

Consider now the case where $\min(v_1, v_2) \leq \frac{b_1}{m}$. Let $b'_i = \min(b'_1, b'_2)$. First, observe that we have that if $b'_i = b'_1$ then $\frac{b_1}{m} \leq \frac{b'_1}{m-1}$, since $b'_i = b'_1$. For $b'_i = b'_2 = b_2 - \frac{b_1}{m}$, we also have that $\frac{b'_2}{m-1} = \frac{b_2 - \frac{b_1}{m}}{m-1} \geq \frac{b_1 - \frac{b_1}{m}}{m-1} \geq \frac{b_1}{m}$. Hence in this range, by Lemma 5.3, it is pareto optimal to allocate all items to the bidder with the highest value, as A_{m-1} indeed does. ∎

Claim 7.12 *A_{m-1} is incentive compatible.*

Proof: Once again we consider the several different cases. Start with the case where $v_1, v_2 > \frac{b_1}{m}$, and suppose player i declares $v'_i > \frac{b_1}{m}$ instead (and is allocated x'_i and pays p'_i). Clearly, $i \neq 1$, as the allocation and payment of player 1 are the same as in A, and A is truthful. Suppose $i = 2$ is better off declaring v'_2: $v_2(x_2) - p_2 < v_2(x'_2) - p'_2$. Observe that in A we have that: $v_2(x_2+1) - (p_2 + \frac{b_1}{m}) < v_2(x'_2+1) - (p'_2 + \frac{b_1}{m})$, a contradiction to the truthfulness of A.

In the case where $\min(v_1, v_2) \leq \frac{b_1}{m}$ player i is not better off declaring $v'_i < \frac{b_1}{m}$, as in this range we essentially conducting a second price auction, which is truthful.

Suppose that $v_1, v_2 > \frac{b_1}{m}$, and that player i declares $v'_i < \frac{b_1}{m}$ instead. Notice that $x'_i = 0$, so i cannot increase his profit from declaring v'_i.

Finally, suppose $\min(v_1, v_2) \leq \frac{b_1}{m}$. Consider player i that declares $v'_i > \frac{b_1}{m}$. Suppose $v_j > \frac{b_1}{m}$, where j is the other player. Observe that if i wins some items, then by Claim 7.9 j has to pay at least $\frac{b_1}{m}$ for every item he wins, which is more than is value. If $v_j < \frac{b_1}{m}$, then we are also in the case of a second price auctions, regardless of what i declares, and this auction is truthful. ∎

Appendix D: Proofs for the Infinitely Divisible Good Setting

Proof: (of proposition 7.2) Pareto-optimality follows directly from proposition 2.4 since in the first two cases the low bidder gets allocated 0, and in the last case, the high bidder has his budget exhausted.

Let us start by looking at the incentives of bidder 1. If $v_2 \leq 1$ then he is faced with exactly two possibilities $x_1 = 1, p_1 = v_2$ and $x_1 = 0, p_1 = 0$ it is clear that he prefers the former only if $v_1 \geq v_2$, which is what happens with the truth. If $v_2 > 1$ then he is faced with two possibilities: either declare some $z \leq v_2$ in which case he gets $x_1 = 0, p_1 = 0$ or declare some $z > v_2$ and get allocated $x_1 = 1/v_2, p_1 = 1$. his utility in the first case is $u_i = 0$ and in the second $u_i = v_1/v_2 - 1$, which is positive iff $v_1 > v_2$ and given to him by the mechanism when telling the truth $z = v_1$.

Now for bidder 2. The case $v_1 \leq 1$ is as before. Otherwise he may declare either $z < v_1$ getting $x_2 = 1 - 1/z, p_2 = \ln z$ or declaring $z \geq v_1$ getting $x_2 = 1, p_2 = 1 + \ln v_1$. In the first case his utility is $u_2 = v_2 - v_2/z - lnz$, which is maximized by saying the truth $z = v_2$ (since we must have $du_2/d_z = 0$ at that point) giving $u_2 = v_2 - 1 - \ln v_2$, and in the second case his utility is $u_2 = v_2 - 1 - \ln v_1$. Clearly the first is better iff $v_2 < v_1$ which is obtained by telling the truth. ∎

Proof: (of proposition 7.5) Let us consider the incentives of one bidder with value v_i when the other bids a fixed value v_j. If $v_j \leq 1$ then bidder i can choose between declaring $z < v_j$ in which case $x_i = 0, p_i = 0$ and thus $u_i = 0$ (the case of tie where $x_i = 1, p_i = v_j$ which still gives $u_i = 0$) to bidding $z > v_j$ in which case $x_i = 1, p_i = v_j$ and thus $u_i = v_i - v_j$. It is clear that the latter is better iff $v_i > v_j$, which happens when bidding the truth. (For now we are ignoring the possibility of declaring $z = v_j$, which we will get to below.)

If $v_j > 1$, then bidder i can choose between declaring $z < v_j$ in which case $x_i = 1/2 - 1/(2v_i^2), p_i = 1 - 1/v_i$ to bidding $z > v_j$ in which case $x_i = 1/2 + 1/(2v_j^2), p_i = 1$. The utility of the former choice is better iff $v_i(1/(2v_i^2) + 1/(2v_j^2)) < 1/v_i$, i.e. exactly when $v_i < v_j$, which is exactly what he gets by bidding the truth.

The only difference between mechanisms B and C is in how they treat $z = v_j$. B treats this case as either $z < v_j$ or $z > v_j$ according to whether i is the first and second bidder. Thus this possibility was already covered by one of the cases we analyzed. C takes the average of these two cases so still it does not produce better results than the higher one. ∎

Proof: (of theorem 7.6) Let us fix a mechanism that satisfies the above properties and reason about it. In the rest of the proof we denote the smaller value by v_i, thus $v_i \leq v_j$.

Step 1: We first handle the case of $v_i \leq 1$. If also $v_j < 1$ then $p_j \leq v_j < 1$ and thus PO implies $x_i = 0$ and $x_j = 1$. By the usual arguments of IC we must have $p_j = v_i$. Now for values $v_j \geq 1$, if $x_j = 1$ then by IC p_j is determined by x_j and thus is $p_j = v_i$. Otherwise $x_i > 0$ and thus by PO $p_j = 1$ but this is a contradiction to IC since declaring a value $v_i < v_j' < 1$ both increases x_j and decreases p_j.

Step 2: We will now show that there exist functions $q(t)$ and $p(t)$ such that whenever $v_i < v_j$ then $x_i = q(v_i)$, $p_i = p(v_i)$, and $x_j = 1 - q(v_i)$, $p_j = 1$. I.e. the low player's value determines the allocation between the two players as well as his own payment, while the high player exhausts his budget. First assume to the contrary that for some $1 < v_i < v_j$, $p_j < 1$, and thus by PO $x_i = 0$, $p_i = 0$, and $x_j = 1$. But then a bidder with $p_j < v_j' < 1 < v_i$ that, according to step 1, gets nothing, would be better off declaring v_j and getting positive utility, in contradiction to IC. Thus $p_j = 1$ whenever $1 < v_i < v_j$. Thus, by IC, for a fixed v_i, different values of v_j must get the same x_j, i.e. x_j depends only on v_i. By PO, $x_i = 1 - x_j$ and thus it also only depends on V_i, and then by IC p_i must be determined uniquely by x_i and thus depends only on v_i.

Step 3: Using IC as usual, we have that for any $1 < t < t' < v_j$: $t(q(t') - q(t)) \leq p(t') - p(t) \leq t'(q(t') - q(t))$. As usual this implies that $dp/dt = t \cdot dq/dt$ or, more precisely, since we do not know that q is differentiable or even continuous, that $p(t) = tq(t) - \int_1^t q(x)dx$, where integrability of q is a direct corollary of its monotonicity. (This already takes into account the boundary condition that for t approaching 1 from above, $q(x)$ must approach 0, as otherwise for the fixed limit $\delta > 0$ we will have that for every value of $v_2 > v_1 > 1$, we will have $x_2 \leq 1 - \delta$, which by IR implies $p_2 < 1$ and thus contradicts PO.)

Step 4: Using IC we have that for $1 < t < v_j < t'$: $tq(t) - p(t) \geq t(1 - q(t')) - 1$ and $t'q(v_j) - p(v_j) \geq t'(1 - q(v_j)) - 1$ Letting t, t' approach v_j we have that $tq(t) - p(t) = t(1 - q(t)) - 1$, i.e. $p(t) = 1 + t(2q(t) - 1)$ for all t except for at the at most countably many points of discontinuity of q.

Step 5: Combining the last two steps we have $1 + t(2q(t) - 1) = tq(t) - \int_1^t q(x)dx$, i.e. $q(t) = 1 - 1/t - (\int_1^t q(x)dx)/t$, except for at most the countably many points of discontinuity of q. The solution to this differential equation, is $q(t) = 1/2 - 1/(2t^2)$, which gives $p(t) = 1 - 1/t$. The uniqueness of solution is implied since if another function satisfies the equation everywhere except for countably many points, then the difference function $d(t)$ would satisfy $d(t) = -(\int_1^t d(x)dx)/t$ everywhere except for countably many points, which only holds for $d(t) = 0$. ∎

Proof: (of theorem 7.7) We first note that by direct scaling of theorem 7.6 we have that that the only anonymous IC+PO mechanism for the case of a publically known budget $b_1 = b_2 = B$ gives $x_i = (1 - B^2/v_i^2)/2$,

$p_i = B(1 - B/v_i)$, $x_j = (1 + B^2/v_i^2)/2$, $p_j = 1$ for the case $1 < v_i < v_j$, and $x_j = 1$, $p_j = v_i$, $x_i = 0$, $p_i = 0$ for the case $v_i < 1$ and $v_i < v_j$.

Let us now assume to the contrary that an annonymous IC+PO auction existed, then for any fixed values of b_1, b_2 it must be identical to the scaled version of mechanism C. Now let us look at a few cases with $v_1 = 2$, $v_2 = 2 + \epsilon$. First let us look at the case $b_1 = b_2 = 1$. The previous theorem mandates that in this case $x_1 = 3/8$, $p_1 = 1/2$ and $x_2 = 5/8$, $p_2 = 1$, (and thus $u_2 = 1/4 + O(\epsilon)$.)

Now let us look at the case where $b_1 = b_2 = 2 - \epsilon$. Again the theorem 7.6 with scaling mandates that $x_1 > 0$ and also $u_1 > 0$.

Now let us look at the case of $b_1 = 1$ and $b_2 = 2 - \epsilon$. If $x_2 < 1$ then, by PO, $p_2 = b_2 = 2 - \epsilon$, and thus $u_2 < 2\epsilon$, which means that player 2 has a profitable lie stating $b_2 = 1$. Thus $x_2 = 1$ and $x_1 = 0$, but then player 1 has a profitable lie stating that $b_1 = 2 - \epsilon$.

∎