

Lecture 9 (Social Choice Cont.)

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1 Impossibility Theorems

N := Set of players/voters.

n := $|N|$.

A := Set of alternatives. $|A| \geq 3$

L := Set of all linear orders over A .

Definition 1 : a **Preference of a voter** is a linear order over A , i.e. a member of L .

Definition 2 : a **Profile** is a member of L^N , I.e. a preference for each voter.

Definition 3 : a **Social Welfare Function** is a function $f:L^N \rightarrow L$.

Definition 4 : a **Social Choice Function** is a function $f:L^N \rightarrow A$.

Notation 1 A preference of a voter is usually notated as $<_i$ (or $>_i$) where i is the voter.

$aR^i b / a >_i b / b <_i a$: all mean 'voter i prefers a over b '.

The result of the function is usually notated as $<$ (or $>$)

We will state and proof :

Arrow's theorem that deals with social welfare functions.

Gibbard-Satterthwaite's theorem that deals with social choice functions.

1.1 Arrow's theorem

Definition 5 A social welfare function f is a **dictatorship** if $\exists i [a > b \iff a >_i b]$.

Theorem 6 Arrow's theorem

f is a social choice function that has the following attributes:

- Unanimity (Pareto-optimality): $[\forall i a \succ_i b] \Rightarrow a \succ b$
- Independence of Irrelevant Alternatives (IIA): $a \succ b$ is a function of only $\{a \succ_i b\}_{i=0}^n$
 $\forall R^N, Q^N$ profiles $[\forall i a R^i b \iff a Q^i b] \Rightarrow [af(R^N)b \iff af(Q^N)b]$

$\Rightarrow f$ is a dictatorship

Comment 1 Pareto-attribute in general: There is no improvement to each of the individuals (or to a subgroup of the individuals) that does not harm any other individual.

This is the weakest notion of optimality in economics.

Lemma 7 (neutrality in pairs)

Definition 8 Neutrality = No prior preference of the function over the alternatives.

$$\forall (\alpha, \beta) \neq (a, b) [\forall i a \succ_i b \iff \alpha \succ_i \beta] \Rightarrow [a \succ b \iff \alpha \succ \beta]$$

(I.e. the aggregation of the voters' preferences over a pair is independent of the pair)

f that satisfy the conditions in Arrow's theorem satisfy neutrality.

Proof of Lemma 7:

Given a profile R^N that satisfy $[\forall i a \succ_i b \iff \alpha \succ_i \beta]$.

With no loss of generality assume $af(R^N)b$.

We will define a new profile Q^N (partially).

$$\forall \text{ voter } i \ a R^i b \Rightarrow \alpha Q^i a Q^i b Q^i \beta$$

$$\forall \text{ voter } i \ b R^i a \Rightarrow b Q^i \beta Q^i \alpha Q^i a.$$

Notice that $[a R^i b \iff a Q^i b]$ and $[\alpha R^i \beta \iff \alpha Q^i \beta]$

$$\text{(Unanimity)} \ [\forall i \alpha Q^i a] \Rightarrow \alpha f(Q^N) a$$

$$\text{(Unanimity)} \ [\forall i b Q^i \beta] \Rightarrow b f(Q^N) \beta$$

$$\text{(IIA)} \ [a R^i b \iff a Q^i b] \Rightarrow [af(R^N)b \Rightarrow af(Q^N)b]$$

$$\text{(Transitivity)} \ [af(Q^N)a \wedge af(R^N)b \wedge bf(Q^N)\beta] \Rightarrow \alpha f(Q^N)\beta$$

$$(IIA) [\alpha R^i \beta \iff \alpha Q^i \beta] \Rightarrow [\alpha f(R^N) \beta \iff \alpha f(Q^N) \beta]$$

Hence $\alpha f(R^N) \beta$

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Proof: (Arrow's theorem)

We'll define a series of profiles $\{R_i^N\}_{i=0}^n$:

$$\begin{array}{cccccc} a\dots a & ba\dots a & \dots & b..ba\dots a & \dots & b..ba & b..b \\ b\dots b & ab\dots b & & a..ab\dots b & & a..ab & a..a \end{array}$$

I.e.

The k^{th} voter's preference in R_i^N is

- $a > b$ if $k > i$
- $a < b$ if $k \leq i$

unanimity \Rightarrow For $R_0^N \rightarrow a > b$. (I.e. a is preferred over b in $f(R_0^N)$)

unanimity \Rightarrow For $R_n^N \rightarrow b > a$.

$\Rightarrow \exists h^* : 1 \leq h^* \leq n$ s.t.

For $R_{h^*-1}^N \rightarrow a > b$

For $R_{h^*}^N \rightarrow b > a$

The only difference between the two profiles is the vote of h^* .

Claim 9 h^* is a dictator

Let α, β, c be three different alternatives in A . (We assumed that $|A| \geq 3$)

We will proof that $[\alpha >_{h^*} \beta \Rightarrow \alpha > \beta]$

Let R^N be a profile in which $\alpha >_{h^*} \beta$

We'll define a new profile Q^N

$$\forall i < h^* \quad \begin{array}{l} \frac{R^N}{\alpha >_i \beta} \Rightarrow \frac{Q^N}{c >_i \alpha >_i \beta} \\ \frac{R^N}{\beta >_i \alpha} \Rightarrow \frac{Q^N}{c >_i \beta >_i \alpha} \end{array}$$

$$\text{for } i = h^* \quad \begin{array}{l} \frac{R^N}{\alpha >_i \beta} \\ \frac{Q^N}{\alpha >_i c >_i \beta} \end{array}$$

$$\forall i > h^* \quad \begin{array}{l} \underline{R^N} \\ \alpha >_i \beta \\ \beta >_i \alpha \end{array} \Rightarrow \begin{array}{l} \underline{Q^N} \\ \alpha >_i \beta >_i c \\ \beta >_i \alpha >_i c \end{array}$$

In Q^N

- $\forall i \geq h^* \alpha >_i c$
 $\Rightarrow \alpha > c$

The preferences over $\{\alpha, c\}$ in Q^N is the same as the preference over $\{a, b\}$ in $R_{h^*-1}^N$ and according to the lemma of neutrality in pairs, the aggregation of a pair is independent in the pair.

- $\forall i \leq h^* c >_i \beta$
 $\Rightarrow c > \beta$

The preferences over $\{c, \beta\}$ in Q^N is the same as the preference over $\{a, b\}$ in R_h^N and according to the lemma of neutrality in pairs, the aggregation of a pair is independent in the pair.

The result preference $>$ is transitive $\Rightarrow \alpha > \beta$.

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1.2 Gibbard-Satterthwaite's theorem

Definition 10 *f is monotone if*

$$\forall i \forall >_i \forall \widehat{>}_i \forall >_{-i} \\ [f(<_i, <_{-i}) = a \neq b = f(\widehat{<}_i, <_{-i})] \Rightarrow [a >_i b \wedge b \widehat{>}_i a]$$

Theorem 11 (*Gibbard-Satterthwaite*)

f is a social choice function

- *f is surjective. $\forall a \in A \exists (<_1, <_2, <_3, \dots, <_n)$ s.t. $f(<_1, <_2, <_3, \dots, <_n) = a$.*
- *f is monotone*

$\Rightarrow f$ is a dictatorship.

Being monotone is equivalent to being resistant to one voter's manipulation.

if f is not monotone then $\exists i \exists >_i \exists \widehat{>}_i \exists >_{-i}$

$$[f(<_i, <_{-i}) = a \neq b = f(\widehat{<}_i, <_{-i})] \wedge [a <_i b \vee b \widehat{<}_i a]$$

- If $[a <_i b]$ then $\widehat{<}_i$ is a worthwhile manipulation for i .
- If $[b <_i a]$ then $<_i$ is a worthwhile manipulation for i .

From the other hand, being monotone states that changing your vote, might only hurt you according to your original vote.

Hence Gibbard-Satterthwaite's theorem states that the only social choice function that cannot be manipulated is dictatorship.

Proof:

We will build a social welfare function F .

For a profile $R^N \forall a, b \in A \ a > b \iff f(Q^N) = a$

where Q^N is a profile defined the following way:

$$\forall i [a Q^i b \iff a R^i b] \wedge [\forall c \notin \{a, b\} \ a Q^i c \wedge b Q^i c] \wedge [\forall c, d \notin \{a, b\} \ c Q^i d \iff c R^i d]$$

claims

1. The output relation is an order.
 - Anti-symmetric
 - Transitive
2. It satisfy unanimity property
3. It satisfy IIA property
4. F is dictatorship $\Rightarrow f$ is dictatorship

(1)+(2)+(3)+Arrow's theorem $\Rightarrow F$ is dictatorship.

F is dictatorship+(4) $\Rightarrow f$ is dictatorship.

Proof: (claim 2 - unanimity)

Let $R^N \in L^N$; $a, b \in A$ s.t. $\forall i \ a >_i b$. We will show that $a F(R^N) b$

Let Q^N be the profile generated by the above process from R^N

$$\forall i \ a Q^i b \wedge [\forall c \notin \{a, b\} \ a Q^i c \wedge b Q^i c] \wedge [\forall c, d \notin \{a, b\} \ c Q^i d \iff c R^i d]$$

We will show that $f(Q^N) = a$ and hence $a F(R^N) b$.

f is surjective $\Rightarrow \exists A^N \in L^N$ s.t. $f(A^N) = a$.

We will gradually lift a in the preferences of the voters and show that the result does not change in the process.

We will define a series of profiles $\{S_i^N\}_{i=0}^n$:

The k^{th} voter's preference in S_i^N is $\begin{cases} \text{his preference in } A^N & \text{if } i > k \\ \text{his preference in } Q^N & \text{if } i \leq k \end{cases}$

- $S_0^N = A^N$
- $S_n^N = Q^N$
- All the voters except k don't change their preference between S_{k-1}^N and S_k^N
- If $f(S_{k-1}^N) = a \neq c = f(S_k^N)$

$\xRightarrow{\text{Monotonicity}}$ $\begin{cases} \text{voter } k \text{ prefer } a \text{ over } c \text{ in } S_{k-1}^N \\ \text{voter } k \text{ prefer } c \text{ over } a \text{ in } S_k^N \end{cases}$

But in S_k^N voter k votes as in Q^N i.e. a is preferred over any other alternative.

Hence $f(S_{k-1}^N) = a \Rightarrow f(S_k^N) = a$

- $f(S_0^N) = f(A^N) = a \Rightarrow f(Q^N) = f(S_n^N) = a$

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Proof: (claim 3 - IIA)

Let $a, b \in A$.

We'll show that for any two profiles R_1^N and R_2^N

$[\forall i \ aR_1^i b \iff aR_2^i b] \Rightarrow [af(R_1^N)b \iff af(R_2^N)b]$

Let Q_1^N, Q_2^N be the profiles generated by the above process from R_1^N and R_2^N respectively.

$aF(R_1^N)b \Rightarrow f(Q_1^N) = a$

We'll define a series of profiles $\{Z_i^N\}_{i=0}^n$

The k^{th} voter's preference in Z_i^N is $\begin{cases} \text{his preference in } Q_1^N & \text{if } i > k \\ \text{his preference in } Q_2^N & \text{if } i \leq k \end{cases}$

- $Z_0^N = Q_1^N$
- $Z_n^N = Q_2^N$
- All the voters except k don't change their preference between Z_{k-1}^N and Z_k^N
- If $f(Z_{k-1}^N) = a \neq c = f(Z_k^N)$

$\xRightarrow{\text{Monotonicity}}$ $\begin{cases} \text{voter } k \text{ prefer } a \text{ over } c \text{ in } Z_{k-1}^N \\ \text{voter } k \text{ prefer } c \text{ over } a \text{ in } Z_k^N \end{cases}$

But the only candidate that might be preferred over a in Z_k^N is b and the preference between a and b does not change between Z_{k-1}^N and Z_k^N .

Hence $f(Z_{k-1}^N)=a \Rightarrow f(Z_k^N)=a$

- $f(Z_0^N)=f(Q_1^N)=a \Rightarrow f(Q_2^N)=f(Z_n^N)=a$

$f(Q_2^N)=a \Rightarrow aF(R_2^N)b$

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Proof: (claim 1)

anti-symmetry

Let $x, y \in A$, $x \neq y$, $R^N \in L^N$.

Let Q^N be the profile generated by the above process from R^N

From the definition we can see that the event $[x > y \wedge y > x]$ is impossible since for the profile Q^N as defined above the event $[f(Q^N)=x \wedge f(Q^N)=y]$ is impossible.

Let A^N be a profile s.t. $f(A^N)=a$ (f is surjective, therefore we can find such profile)

We'll define a series of profiles $\{Z_i^N\}_{i=0}^n$

The k^{th} voter's preference in Z_i^N is $\begin{cases} \text{his preference in } A^N & \text{if } i > k \\ \text{his preference in } Q^N & \text{if } i \leq k \end{cases}$

- $Z_0^N = A^N$
- $Z_n^N = Q^N$
- All the voters except k don't change their preference between Z_{k-1}^N and Z_k^N
- If $f(Z_{k-1}^N)=x$, $d=f(Z_k^N)$, $x \in \{a, b\}$, $d \notin \{a, b\}$
 $\begin{matrix} \text{Monotonicity} \\ \Rightarrow \end{matrix} \begin{cases} \text{voter } k \text{ prefer } x \text{ over } d \text{ in } Z_{k-1}^N \\ \text{voter } k \text{ prefer } d \text{ over } x \text{ in } Z_k^N \end{cases}$
 x is either a or b, but no $d \notin \{a, b\}$ is preferred over x in Z_k^N by k.
Hence $f(Z_{k-1}^N) \in \{a, b\} \Rightarrow f(Z_k^N) \in \{a, b\}$
- $f(Z_0^N)=f(A^N)=a \Rightarrow f(Q^N)=f(Z_n^N) \in \{a, b\}$

Hence $[x > y \vee y > x]$

transitivity

Let $a, b, c \in A$ three different alternatives, $R^N \in L^N$.

Assume that $[a > b \wedge b > c \wedge c > a]$ and we will come to contradiction.

We will define a new profile A^N :

$$\forall i [xA^i y \iff [\begin{array}{l} x \in \{a,b,c\} \wedge y \notin \{a,b,c\} \vee \\ x \in \{a,b,c\} \wedge y \in \{a,b,c\} \wedge xR^i y \vee \\ x \notin \{a,b,c\} \wedge y \notin \{a,b,c\} \wedge xR^i y \end{array}]]$$

Let B^N be a profile s.t. $f(B^N)=a$ (f is surjective, therefore we can find such profile)

We'll define a series of profiles $\{Z_i^N\}_{i=0}^n$

The k^{th} voter's preference in Z_i^N is $\begin{cases} \text{his preference in } B^N & \text{if } i > k \\ \text{his preference in } A^N & \text{if } i \leq k \end{cases}$

- $Z_0^N = B^N$
- $Z_n^N = A^N$
- All the voters except k don't change their preference between Z_{k-1}^N and Z_k^N
- If $f(Z_{k-1}^N)=x$, $d=f(Z_k^N)$, $x \in \{a,b,c\}$, $d \notin \{a,b,c\}$
 $\begin{array}{l} \text{Monotonicity} \\ \Rightarrow \end{array} \begin{cases} \text{voter } k \text{ prefer } x \text{ over } d \text{ in } Z_{k-1}^N \\ \text{voter } k \text{ prefer } d \text{ over } x \text{ in } Z_k^N \end{cases}$
 x is a , b , or c , but no $d \notin \{a,b,c\}$ is preferred over x in Z_k^N by k .
Hence $f(Z_{k-1}^N) \in \{a,b,c\} \Rightarrow f(Z_k^N) \in \{a,b,c\}$
- $f(Z_0^N)=f(B^N)=a \Rightarrow f(A^N)=f(Z_n^N) \in \{a,b,c\}$

With no loss of generality $f(A^N)=a$.

We will define a new profile C^N :

$$\forall i [xC^i y \iff [\begin{array}{l} x \in \{a,c\} \wedge y \notin \{a,c\} \vee \\ x \in \{a,c\} \wedge y \in \{a,c\} \wedge xR^i y \vee \\ x \notin \{a,c\} \wedge y \notin \{a,c\} \wedge xR^i y \end{array}]]$$

(C^N is generated by lowering b in A^N)

F is IIA and hence $cF(C^N)a$. ($cF(R^N)a$)

$\Rightarrow f(C^N)=c$. (a,c are the two top alternatives for each of the voters in C^N)

We'll define a series of profiles $\{Z_i^N\}_{i=0}^n$

The k^{th} voter's preference in Z_i^N is $\begin{cases} \text{his preference in } A^N & \text{if } i > k \\ \text{his preference in } C^N & \text{if } i \leq k \end{cases}$

- $Z_0^N = A^N$

- $Z_n^N = C^N$
 - All the voters except k don't change their preference between Z_{k-1}^N and Z_k^N
 - If $f(Z_{k-1}^N) = a \neq d = f(Z_k^N)$
 - \Rightarrow *Monotonicity* $\left\{ \begin{array}{l} \text{voter k prefer a over d in } Z_{k-1}^N \\ \text{voter k prefer d over a in } Z_k^N \end{array} \right.$
- But for every alternative d:
- [voter k prefer a over d in Z_{k-1}^N]
 $\Rightarrow [d \in A \setminus \{a, b, c\}]$
 \Rightarrow [voter k prefer d over a in Z_k^N]
Hence $f(Z_{k-1}^N) = a \Rightarrow f(Z_k^N) = a$
- $f(Z_0^N) = f(A^N) = a \Rightarrow f(C^N) = f(Z_n^N) = a$

We got that $f(C^N) = a \wedge f(C^N) = c$.

contradiction. ■

Proof: (claim 4)

If F is a dictatorship

\exists a voter i (with no loss of generality $i=1$) s.t. $\forall a, b \in A \ a <_1 b \Rightarrow a < b$.

Let R^N be a profile and a an alternative s.t. $\forall b \in A \setminus \{a\} \ a >_1 b$

Assume that $f(R^N) = c \neq a$.

We will look on the profile Q^N that is defined by

$\forall i \ [a Q^i c \iff a R^i c] \wedge [\forall d \notin \{a, c\} \ a Q^i d \wedge c Q^i d] \wedge [\forall d, e \notin \{a, c\} \ d Q^i e \iff d R^i e]$

$f(Q^N) = a$ since F is a dictatorship, hence $a F(R^N) c$

We'll define a series of profiles $\{Z_i^N\}_{i=0}^n$

The k^{th} voter's preference in Z_i^N is $\left\{ \begin{array}{ll} \text{his preference in } Q^N & \text{if } i > k \\ \text{his preference in } R^N & \text{if } i \leq k \end{array} \right.$

- $Z_0^N = Q^N$
- $Z_n^N = R^N$
- All the voters except k don't change their preference between Z_{k-1}^N and Z_k^N

- If $f(Z_{k-1}^N)=a \neq d=f(Z_k^N)$
 $\xrightarrow{\text{Monotonicity}} \left\{ \begin{array}{l} \text{voter } k \text{ prefer } a \text{ over } d \text{ in } Z_{k-1}^N \\ \text{voter } k \text{ prefer } d \text{ over } a \text{ in } Z_k^N \end{array} \right.$

But no alternative is preferred over a for any voter in R^N and hence k does not prefer d over a in Z_k^N .

Hence $f(Z_{k-1}^N)=a \Rightarrow f(Z_k^N)=a$

- $f(Z_0^N)=f(Q^N)=a \Rightarrow f(R^N)=f(Z_n^N)=a$

We got that $f(R^N)=a \wedge f(R^N)=c$.

contradiction. ■

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2 Mechanism Design

We would like to find a preference aggregation mechanism that is manipulation-resistant. From Gibbard-Satterthwaite's theorem we see that one cannot find such not-trivial mechanism. We will explore three ways to 'bypass' the theorem

- Exploring the computational difficulty to manipulate the mechanism.
- Limiting the possible preferences of the voters.
- Changing the preferences to be cardinal and not ordinal (Add money to the mechanism).

2.1 Exploring the computational difficulty to manipulate the mechanism

$f: L^N \rightarrow A$ - a social choice function

The constructive manipulation problem:

Input: $\langle_{-i}, c \in A$.

Output: Whether there exists \langle_i s.t. $f(\langle_i, \langle_{-i})=c$

When A is small the above problem is equivalent to the problem of finding a preference for i that will improve the result comparing to the result of voting sincerely.

We will show that for several social choice functions this problem is NP-complete.

In all these examples we will prove the hardness of the worst-case scenario and not the average-case scenario that might seem more suitable.

(Finding a social choice function that can be proved to be hard on the average-case is an open problem)

social choice functions that are easy to manipulate

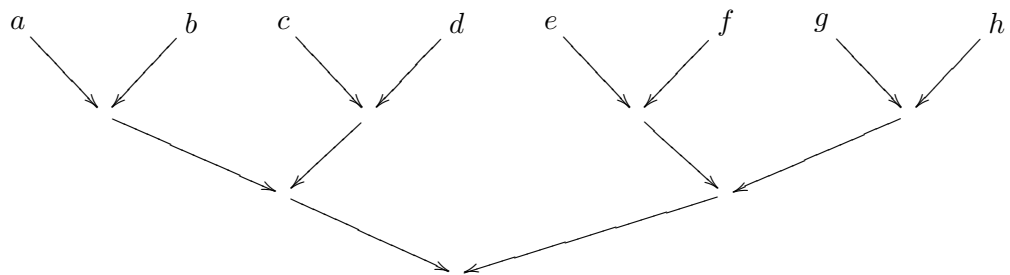
Plurality chooses the alternative that is the first preference for the maximal number of voters with some tie-breaker (e.g. lexicographical)

$$f(R^N) = \min \operatorname{argmax}_a \|\{i : i \text{ prefers } a \text{ over any other alternative}\}\|$$

The only possible manipulation to support an alternative c is placing it on top of the preference. Hence it is an easy task to find whether a manipulation to support c exists. (Only one option should be checked)

CUP The mechanism consists of a binary tree with the alternatives in its leaves. By induction starting from the leaves, in each node one alternative is chosen between the alternatives of the children using plurality vote. The alternative chosen at the root is the mechanism's result.

Example:



This function too has a manipulation that takes polynomial time. (This manipulation is described in exercise 2)

Borda Each voter's preference is translated to scoring mechanism in which each alternative a gets $\sum_{i \in N} \|\{b \in A : a >_i b\}\|$ points.

An alternative with the maximal number of points is chosen (with some tie breaker)

If there is a manipulation to support c , then the following manipulation will work too:

- Giving c n points
- Giving its 'strongest' competitor 1 point.
- Giving its second 'strongest' competitor 2 points.
- \vdots

I.e. each alternative besides c will get from voter i x points where x is its location in the aggregated preference without i and without c .

Therefore i can check only this manipulation to find whether a manipulation exists.

social choice functions that are hard to manipulate

STV Single Transferable Vote description:

1. Every voter votes for his preferable alternative.
2. The alternative that got the minimal number of votes is deleted.
3. Each voter that voted for the deleted alternative transfers his vote to a new alternative.
4. This process continues until all the voters agree on an alternative.

(Such a phase will come after at most $(|A|-1)$ deletions)

Manipulate the STV function is an NP-complete problem.

Plurality after one iteration of CUP There is a division to pairs of the alternatives. From each pair one alternative is chosen using the plurality voting systems. (As in the first phase of CUP)

From the alternatives that were chosen in the previous iteration one alternative is chosen using plurality.

Theorem 12 *Finding a manipulation for the above mechanism is NP-hard*

Proof:

We will construct a reduction from the max-cut problem which is known to be NP-complete.

The max-cut problem:

Input: A graph $G=(V,E)$ and a natural number k .

Output: Whether there exists a cut of size at least k in G .

The reduction:

Given a graph and a parameter k we will build a voting problem.

Remark With no loss of generality, assume k is even. If k is odd then:

G has a cut of size at least $k \iff \tilde{G}$ has a cut of size at least $2k$

where \tilde{G} is two copies of G .

Remark For the simplicity of the notation an edge between two nodes v_1 and v_2 will be notated by only one of the names (v_1,v_2) , (v_2,v_1) . Therefore the parametric name $e=(x,y)$ has only one possible instantiation.

Alternatives A :

- Two alternatives for each node v : v_L, v_R .
- Four other alternatives $c, c', d,$ and d' .

Other players' preferences R_{-i} (partial description):

Let m be an even number larger than $2|E|$

- \forall edge $e=(v,u)$ we will have two voters v_e, u_e
 1. v_e preference satisfy
 - $v_L > u_R > c > v_R > u_L > x > c' > d' \forall x \notin \{v_L, u_R, c, v_R, u_L, c', d'\}$
 - $\forall w \notin \{v,u\} w_L > w_R$
 2. u_e preference satisfy
 - $v_R > u_L > c > v_L > u_R > x > c' > d' \forall x \notin \{v_L, u_R, c, v_R, u_L, c', d'\}$
 - $\forall w \notin \{v,u\} w_R > w_L$
- m voters : c_1, \dots, c_m
 c_i preference satisfy
 1. $c > x \forall x \neq c$
 2. for every node v $\begin{cases} v_R > v_L & \text{if } i \text{ is odd} \\ v_L > v_R & \text{if } i \text{ is even} \end{cases}$
- $(m+k)$ voters: d_1, \dots, d_{m+k} d_i preference satisfy
 1. $d > x \forall x \neq d$
 2. for every node v $\begin{cases} v_R > v_L & \text{if } i \text{ is odd} \\ v_L > v_R & \text{if } i \text{ is even} \end{cases}$

In the first phase (CUP) the voters will be coupled to pairs in the following way:

- $\forall v : v_L$ will be coupled with v_R .
- c' will be couples with c
- d' will be couples with d

In the second phase the remaining alternatives are chosen using plurality (In case of a tie between c and d , c is preferred)

The manipulation problem: Given R_{-i} , does i has a preference so $f(R_i, R_{-i})=c$?

Notice that the build of the manipulation problem can be done in polynomial time.

Claim 13 For every node v exactly half of the voters in R_{-i} prefer v_L over v_R

- Voters of type e_w :
 - For any edge $e=(v, x)$ e_v prefers v_L over v_R
 e_x prefers v_R over v_L
 - For any edge $e=(x, v)$ e_x prefers v_R over v_L
 e_v prefers v_L over v_R
 - For any other edge (x,y) e_x prefers v_L over v_R
 e_y prefers v_R over v_L
- Voters of type c_i : The odd voters prefer v_R over v_L
 The even voters prefer v_L over v_R
 The number of voters is m which is even
- Voters of type d_i : The odd voters prefer v_R over v_L
 The even voters prefer v_L over v_R
 The number of voters is $(m+k)$ which is even

■

Every voter prefers c over c' and d over d' . Therefore c and d will pass to the second phase.

All m c_i voters will vote in the second phase to c .

All $(m+k)$ d_i voters will vote in the second phase to d .

Since the number of the remaining voters is at most $2|E| < m$, the result of the mechanism will be in $\{c, d\}$.

Every voter of type e_x (e is an edge between v and u) will vote in the second phase to one of the following: c, v_L, v_R, u_L , or u_R .

For a preference R_i : The preference (vote) of i defines a cut of G (noted (S, S^C)) in the following way:

$$S = \{v \in V : v_L >_i v_R\}$$

Based on claim 13, i is a tie breaker in the first phase. I.e.:

v_L ($v \in V$) is chosen in the first phase $\iff v \in S$.

v_R ($v \in V$) is chosen in the first phase $\iff v \notin S$.

Claim 14 i can manipulate the mechanism to choose c iff there is a cut in G of size at least k .

Hence the manipulation problem is at least as hard as the max-cut problem.

i can manipulate the mechanism to choose c by voting R_i

\iff At least k voters of type e_v voted for c in the second phase

\iff There is at least k pairs $\langle v_L, u_R \rangle$ s.t.

- There is an edge in G between v and u .
- v_L was chosen in the first phase between v_L, v_R
- u_R was chosen in the first phase between u_L, u_R

\iff There is at least k pairs $\langle v_L, u_R \rangle$ s.t.

- There is an edge in G between v and u .
- $v \in S$
- $u \notin S$

\iff (S, S^C) is a cut of size at least k .

Therefore i can manipulate the mechanism \iff there exists a cut of size at least k in G .

■