Topics on the Border of Economics and Computation	December 11, 2005
Lecture 7	
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1 Nash's Theorem

We begin by proving Nash's Theorem about the existance of a mixed strategy equilibrium in every finite game. The proof relies on one of the fixed point theorems - Brouwer's fixed point theorem.

Theorem 1 *Brouwer's fixed point theorem: If C is a compact (closed and bounded) convex set, and $f : C \to C$ is a continuous function, then there exists $c \in C$ for which $f(c) = c$.*

We now use Brouwer's fixed point theorem to prove Nash's theorem.

Theorem 2 *Nash's theorem: Every finite game has a mixed strategy equilibrium.*

Proof Idea: Let $u_i(s_1, ..., s_n)$ denote player i's utility when the players choose strategies $s_1, ..., s_n$. We denote player i's strategic choice x_i and the other players' choices x_{-i}. We define the best response of player i to x_{-i}, $BR_i(x_{-i})$, as the x_i for which for all other x'_i we have $u_i(x_i, x_{-i}) > u_i(x'_i, x_{-i})$. We have defined $x_1, ...x_n$ to be a Nash equilibrium if each of the x_is is the best response to x_{-i}. Let $\Delta(S_i)$ be the mixed extension of player i's strategies. We define a transformation from $\Delta(S_1) \times \Delta(S_2) \times ... \times \Delta(S_n)$ onto itself, that is the best response transformation: $(x1, ..., x_n) \to BR(x_1), ..., BR(x_n)$. The fixed point of this transformation is a Nash equilibrium. The set is compact and convex. Had this transform been a function we could use Brouwer's theorem to show that a fixed point (Nash equilibrium) exists. However, there may be more than one best response, and the transform is now continous. Thus, we would denote $\hat{x_i}$ a certain adjustment of x_i toward the best response. If no adjustment is required, we already have a best response. ■

Proof: We define a continuous function from $\Delta(S_1) \times \Delta(S_2) \times ... \times \Delta(S_n)$ onto itself: $(x_1, ..., x_n) \to \hat{x_1}, ..., \hat{x_n}$. We denote x_i^j the weight of pure strategy j in the distribution x_i.

We denote $c_i^j = c_i^j(x_1, ..., x_n) = u_i(j, x_{-i}) - u_i(x_i, x_{-i})$ the payoff of player i when switching from x_i to pure strategy j.

c_i^j is continuous for all i, j, since u_i is linear in x_i. At the equilibrium point, every $c_i^j < 0$.

We'd like to define $\hat{x}_i^j = x_i^j + c_i^j$, but we have to normalize to get a probability distribution.

We denote $c_i^{j+} = \max(0, c_i^j)$. c_i^{j+} is also continuous as the maximum of two continuous functions. We note that for every i, $\sum_i c_i^{j+} x_i^j = 0$ as $c_i^j = u_i(j, x_{-i}) - u_i(x_i, x_{-i})$ and we are summing this over all js.

We define $\hat{x}_i^j = \frac{x_i^j + c_i^{j+}}{1+\sum_j c_i^{j+}}$. We note that for every i, $\sum_j \hat{x}_i^j = 1$.

As noted before, the c_i^js are continuos, the u_is are continuous as a linear function. Also, the maximum, sums and divisions of continuous functions are also continuous. Thus, The function f is continuous. We now use Brouwer's fixed point theorem to show that f has a fixed point. In such a point, for every i, $x_i = \hat{x}_i$. We show that if the games is in a Nash equilibrium, we have $c_i^{j+} = 0$ for every i, j. If the game is in a Nash equilibrium and yet there exists some $c_i^{j+} > 0$, then $x_i^j = \hat{x}_i^j > 0$. Since $\sum_j x_i^j c_i^j = 0$ there must exist some j' that $c_i^{j'} < 0$, so $c_i^{j'+} = 0$. Then $x_i^{j'} = \hat{x}_i^{j'} = \frac{x_i^{j'}}{\sum_j c_i^{j+}}$. But this is a contradiction, since $1 + \sum_j c_i^{j+} > 1$ and we get that $x_i^{j'}$ is equat to itself divided by a number greater than 1. ■

2 Computational hardness

Note that the proof in the previous section only shows a Nash equilibrium in mixed strategies exists, but does not show how to find that equilibrium. The proof of Brouwer's fixed point theorem contains a parity argument. The Nash equilibrium problem with two players is complete in PPAD (polinomail parity argument directed).

Finding any Nash equilibrium is a PPAD-complete. Almost any condition on the Nash equilibrium to find makes the problem NP-complete.

Theorem 3 *Finding the Nash equilibrium that maximizes the social welfare ($u_1 + u_2$) is NP-complete.*

In this problem we are given the utility matrixes of players 1 and 2, and as an output we need to give the mixed strategies which are the Nash equilibrium that maximizes social welfare.

To proof the theorem we use a reduction from Bi-Clique.

The Bi-Clique problem: we are given a biparite graph and a number k. We need to decide whether there exist 2 sets of k vertices (one in each side) that hold between them all the edges in the graph.

Sketch of Proof: To reduce a Bi-Clique $G =< V, E >, k$ problem to a maximal social wellfare nash equilibrium problem, we build the payment matrixes. Let L be the set of the

Table 1: The reduced game for Bi-Clique

	R	L
L	$\begin{matrix}(1,1) & \text{if} & (r,l) \in E \\ (0,0) & \text{if} & (r,l) \notin E\end{matrix}$	$\begin{matrix}(-k,k) & & 0 \\ & \ddots & \\ 0 & & (-k,k)\end{matrix}$
R	$\begin{matrix}(k,-k) & & 0 \\ & \ddots & \\ 0 & & (k,-k)\end{matrix}$	0

vertices in the left side of the biparite graph, and R the set of vertices in the right side. The payment matrixes are composed of 4 submatrixes - top left, top right, bottom left and bottom right. In the top left submatrix, we have a row for every vertex in L and colomn for every vertex in R. If the edge $(l,r) \in E$ the payments are $(1,1)$. If $(l,r) \notin E$ the payments are $(0,0)$. In the top right submatrix, we have a row for every vertex in R and colomn for every vertex in L. The payments are always $(-k,k)$ in the diagonal, and $(0,0)$ anywhere else. In the bottom left submatrix, we have a row for every vertex in R and colomn for every vertex in L. The payments are always $(k,-k)$ in the diagonal, and $(0,0)$ anywhere else. In the bottom right submatrix, we have a row for every vertex in L and colomn for every vertex in R. The payments are $(0,0)$ everywhere in that submatrix.

We show that in the game described by the built matrix there is a Nash equilibrium with a total utility of $u_1 + u_2 \geq 2$ if and only if there is a k-bi-clique in the original graph.

In the first direction, we can show that when we choose the strategies that are the vertices of the bi-clique, we get a utility of exactly 2. We can also show that this is a Nash equilibrium: for example, first player can only loose by deviating on the left side.

In the opposite direction, we consider a Nash equilibrium with a utility of at least 2. The strategies with positive probability define vertices in the original graph. We can show that these are a bi-clique. We can show that if there is any weight in the mixed strategy in the bottom, the total utility would be less than 2. So, if the total utility is at least 2, all the weight is in the top left submatrix. We then note that if even one strategy is missing, we get a total utility which is less than 2. In order to complete the proof we show that if there are less than k chosen strategies, the weight of at least one chosen strategy is at least $\frac{1}{k}$, and look at what we can do on the bottom left submatrix. ■

3 Correlated equilibrium

We start by giving the game of Chicken as an example. Consider a head-to-head race between to players. In this game two individuals are challenging each other to a contest

Table 2: The game of Chicken

	Chicken	Dare
Chicken	(8,8)	(1,9)
Dare	(9,1)	(0,0)

where each can either dare or chicken out. If one is going to Dare, it is better for the other to chicken out. But if one is going to chicken out it is better for the other to Dare. This leads to an interesting situation where each wants to dare, but only if the other might chicken out.

In this game, there are three Nash equilibria. The two pure strategy Nash equilibria are (D, C) and (C, D).

In order to find the mixed strategy equilibrium we denote by p the probability of dare. In the equilibrium we have $8p + 1(1-p) = 9p$, so $p = 0.5$. In this case we have $u_1 = 4.5$. However, both players could get $(8, 8)$ if they both chicken out.

Now consider a trusted third party that draws one of three cards labeled: (C, C), (D, C), and (C, D). He draws each of the cards with equal probability. After drawing the card the third party informs the players of the strategy assigned to them on the card *but not the strategy assigned to their opponent.*

We show that listening to the trusted third party is an equilibrium. Suppose a player is assigned D, he would not want to deviate supposing the other player played their assigned strategy since he will get 9 (the highest payoff possible). Suppose a player is assigned C. Then the other player will play C with probability 0.5 and D with probability 0.5. The expected utility of Daring is $0.5 \cdot 0 + 0.5 \cdot 9 = 4.5$. The expected utility of chickening out is $0.5 \cdot 8 + 0.5 \cdot 1 = 4.5$. So, there is not incentive not to do as the third party has instructed.

Since neither player has an incentive to deviate, this is a correlated equilibrium. Interestingly, the expected payoff for this equilibrium is $\frac{1}{3} \cdot 8 + \frac{2}{3} \cdot 5 = 6$ which is higher than the expected payoff of the mixed strategy Nash equilibrium.

Let us denote the the probability the trusted third party chooses $(C, C), (C, D), (D, C), (D, D)$ by a, b, c, d. The condition for a correlated equilibrium is that given the trusted third party's information, each player has no incentive to deviate. Thus, we get several equations. $9 \cdot \frac{c}{c+d} + 0 \cdot \frac{d}{c+d} \geq 8\frac{c}{c+d} + 1\frac{d}{c+d}$ and we get $b \geq a$. Similarly we get $c \geq a$. On the other side we get $8 \cdot \frac{a}{a+b} + 1 \cdot \frac{b}{a+b} \geq 9\frac{a}{a+b} + 0\frac{b}{a+b}$ and we get $b \geq a$, and similarly $b \geq d$.

So, the neccessary and sufficient condtions for a correlated equilibrium in this game are:

$a + b + c + d = 1$, $a, b, c, d \geq 0$, $b \geq d$, $c \geq a$, $c \geq d$, $b \geq a$.

Since all the conditions are linear, it is easy to find the optimal probabilities for the trusted third party, which maximize the expected social welfare, by solving a linear program.

In the next lesson we would give the formal definition of correlated equilibrium.