

## Lecture 11

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## 1 Mechanism Design

In this section we will discuss a few problems and solutions related to Mechanism Design.

Let's look at the following problem. There is a set  $A$ .  $L$  is a set of all full orders on  $A$ . There are  $n$  players, where each player  $i$  has preference on  $A$ ,  $<_i \in L$ . The society should choose together member of  $A$ , based on the players' preferences,  $f : L^n \rightarrow A$ . The players' preferences are private, unknown to the builder of the system. The behavior of the players is strategic, i.e. once the rule for extracting the function  $f$  is set, a player can choose his preference on  $A$  according to some strategy for determining the result of function  $f$ . It's possible to add weights to the preferences of the players.

We shall see that it is possible to construct a system representing an "intelligent" function  $f$ , despite the manipulations of the players.

**Example 1** *There is an auction where two players, Alice and Bob, are competing for a single item. Alice values it at  $a$ , and Bob - at  $b$ .  $A = \{\text{Alice wins}, \text{Bob wins}\}$  Here the players' preferences have weights: a number representing how much the player wants to win. Our task as builders of the system is to sell the item to the player who values the item more, i.e. who can propose more money for it.*

Suggestion (Vickrey): sell the item to the player proposing more money, and let him pay as much as the other player proposed.

**Theorem 1** *This method is truthful, i.e. provides that in spite of strategic behavior of the players, no one can gain by lying (not telling the real value).*

**Proof:** As we said, Alice values the item at  $a$ , and Bob values it at  $b$ . Suppose that the players said  $a'$  and  $b'$  respectively. We will prove that Alice cannot gain by telling  $a' \neq a$  (and the prove for Bob is the same). We consider two cases:

Case 1:  $b' < a$ . For each value  $a'$  such that  $a' > b'$ , Alice wins and pays  $b'$  (exactly the same as if she would say  $a$ ). If she says  $a' < b'$ , then she will not win, hence it is better for her to say  $a$ .

Case 2:  $b' > a$ . For each value of  $a'$  so that  $a' < b'$ , Alice would not win the bid (exactly as if she would tell  $a$ ). And if she says  $a' > b'$ , then she would win and have to pay  $b'$  which is more than the item values for her. ■

Note that this method is used nowadays in electronic agents auctions.

**Example 2** *Public project: the government wants to build a bridge that costs \$100M. Every citizen  $i$  would gain from this bridge  $w_i$ . We would like to check whether it is worth to build the bridge, i.e. whether  $\sum w_i > \$100M$ .*

Proposed solutions:

- 1) No one pays. This solution is not good enough: everyone would say  $w_i$  very high, like \$101M, for making the bridge to be built, and so the bridge would be built even if  $\sum w_i < \$100M$ .
- 2) Everyone says his  $w_i$ , and if the bridge is built, everyone pays \$100. In this case, if there is any one with  $w_i > \$100$ , he would say \$101M, and will still pay \$100. Again, we can't check here if  $\sum w_i > \$100M$ .
- 3) Everyone will pay the amount that he said. In this case everyone will say 0 for not financing for the rest the bridge building.

Clark tax: Let's say that the project building costs  $x$ . Everyone tells his  $w_i$ . If  $\sum w_i > x$  then the project is built, and every citizen  $i$  pays  $x - \sum_{j=1, j \neq i}^n w_j$ , i.e. the minimal price that  $i$  should say in order to build the project.

For example, project costs \$200. There are 4 people which say their  $w_i$  : 100, 80, 40, 50. Then the payment would be (in the same order): 30, 10, 0, 0.

**Theorem 2** *Clark tax method provides that for each player it is worth to tell the truth.*

**Proof:** Let  $1 \leq i \leq n$ . The benefit of  $i$  is  $w_i$ . Suppose that each  $1 \leq j \leq n$  said  $w'_j$ . Denote  $t_i = \sum_{j \neq i} w'_j$ . Clark tax scheme imposes that if the project is built, then  $i$  should pay  $x - t_i$ .

Case 1:  $w_i \geq x - t_i$ . If  $i$  says  $w'_i \geq x - t_i$ , then anyway (as with  $w'_i = w_i$ ) he will pay  $x - t_i$ . And if he says  $w'_i < x - t_i$ , then the project won't be built because then  $\sum_{j=1}^n w'_j < x$ , and he will not benefit.

Case 2:  $w_i < x - t_i$ . If  $i$  will say  $w'_i < x - t_i$  then  $\sum_{j=1}^n w'_j = w'_i + t_i < x$  and the project is not executed (the same as if he would say  $w_i$ ). And if he says  $w'_i \geq x - t_i$ , then the project would be executed and  $i$  will have to pay  $x - t_i > w_i$ . So in any case it is better for  $i$  to tell the correct  $w_i$ . ■

The advantage of this scheme is that it encourages people to tell their correct value for the project. The disadvantage: it is not rewarding for the project: if  $\sum_{i=1}^n w_i > x$  then the payments do not cover the investment for the project.

## 2 Games with partial information

In this section we will define pre-bayesian game with partial information, and afterwards discuss an example of such game.

**Definition 3** *Pre-bayesian game with partial information (PBGPI) consists of  $n$  players.  $A_i$  are the possible actions of player  $i$ .  $\Theta$  is a set of all possible worlds of a game. The utility of  $i$  is  $u_i : \Theta \times A_1 \times \dots \times A_n \rightarrow \mathbb{R}$ . The “signal” of  $i$  is  $s_i : \Theta \rightarrow S_i$ . [ $s_i(\theta)$  indicates all the possible worlds of specific game based on  $i$ ’s knowledge: all the  $\theta$  that give the same  $s_i(\theta)$ ].  $a_i : S_i \rightarrow A_i$  is function indicating the action that  $i$  would take after getting the signal  $s_i(\theta)$ .*

**Definition 4**  $a_i(\cdot)$  are in ex-post-Nash equilibrium if for all  $\theta \in \Theta$ ,  $a_i(s_i(\theta))$  are in Nash equilibrium in game defined by  $\theta$ .

This means that also after we know the exact world which we are in, no player would like to change his action. Usually the ex-post-Nash equilibrium does not exist.

### Example 3

	$\theta_1$		$\theta_2$
$8,8$	$1,-1$	$-1,-1$	$0,0$
$-1,1$	$0,0$	$0,0$	$10,10$

This example shows game with two possible worlds,  $\theta_1$  and  $\theta_2$ . In each entry the first number indicates the utility of the “rows” player, and the second is the utility of the “columns” player. The only Nash equilibrium in  $\theta_1$  is in entry  $(8,8)$ , and in  $\theta_2$  is  $(10,10)$ . Hence, if  $s_i(\theta_1) = s_i(\theta_2)$ , there is no ex-post-Nash equilibrium in this example.

**Example 4** Let’s return to our example of public auction with two bidders, Alice and Bob. Alice values the item at  $a$ , and Bob - at  $b$ . This is example of PBGPI with the following parameters:  $\Theta = \mathbb{R}^{+2}$ ,  $(a,b) = \theta \in \Theta$ ,  $A_i = \mathbb{R}^+$

$$u_{Alice}((a,b), a', b') = \begin{cases} a - b' & a' > b' \\ 0 & a' \leq b' \end{cases} \quad u_{Bob}((a,b), a', b') = \begin{cases} b - a' & b' > a' \\ 0 & b' \leq a' \end{cases}$$

$$s_{Alice}(a,b) = a, \quad s_{Bob}(a,b) = b$$

$a_{Alice}(a)$ : the strategy of Alice in the game

$a_{Bob}(b)$ : the strategy of Bob in the game

**Corollary 5**  $a_{Alice}(a) = a$ ,  $a_{Bob}(b) = b$  for all  $a, b \in \mathbb{R}^+$ , are in ex-post-Nash equilibrium.

**Proof:** We already proved it in Section 1, Theorem 1. ■